Dynamic Transitions in a Shear Flow Model

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This talk is based on the paper "Dynamic Transitions and Baroclinic Instability for 3D Continuously Stratified Boussinesq Flows" joint with Shouhong Wang from Indiana University, Bloomington. You can find this paper on arXiv.

Baroclinic Instability

Baroclinic instability is one of the most important geophysical fluid dynamical instability, and plays a crucial role in understanding the dominant mechanism shaping the cyclones and anticyclones that dominate weather in mid-latitudes, as well as the mesoscale eddies that play various roles in oceanic dynamics and the transport of tracers.

The non-dimensional equations describing the baroclinicity for continuously stratified Boussinesq flow is given by

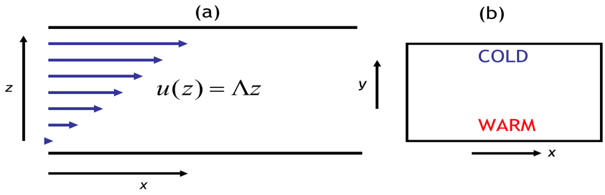
$$\begin{aligned}
&\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathrm{Pr}\Delta\mathbf{u} + \frac{1}{\mathrm{Ro}}\vec{e}_3 \times \mathbf{u} + \nabla p + \frac{1}{\mathrm{Fr}^2}\rho\vec{e}_3 = 0,\\
&\rho_t + (\mathbf{u}\cdot\nabla)\rho - \Delta\rho = 0,\\
&\nabla\cdot\mathbf{u} = 0.
\end{aligned}$$

$\mathbf{u}(x,t) \in \mathbb{R}^3$ is the velocity field, $p(x,t) \in \mathbb{R}$ is the pressure, $\rho \in \mathbb{R}$ is the density (temperature), $x \in \mathbb{R}^3$, $t \geq 0$.

$$\begin{aligned}
&\mathrm{Pr} = \text{Prandtl number} > 0,\\
&\mathrm{Ro} = \text{Rossby number} > 0,\\
&\mathrm{Fr} = \text{Froude number} > 0.
\end{aligned}$$

Steady State Solution

$$\mathbf{u}_{ss} = (\Lambda z, 0, 0), \qquad \rho_{ss} = \frac{\mathrm{Fr}^2 \Lambda}{\mathrm{Ro}} y.$$



Zonal wind varies linearly with altitude.

Aim. Study the stability and transitions of this flow.

Taking the deviations from the basic state

$$\mathbf{u}' = \mathbf{u} - \mathbf{u}_{ss}, \quad p' = p - p_{ss}, \quad \rho' = \rho - \rho_{ss},$$

and dropping the primes, we deduce the following equations for the deviations:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \Lambda z \frac{\partial \mathbf{u}}{\partial x} + \Lambda(\mathbf{u} \cdot \vec{e}_3)\vec{e}_3 - \mathrm{Pr}\Delta\mathbf{u} + \frac{1}{\mathrm{Ro}}\vec{e}_3 \times \mathbf{u} + \nabla p + \frac{1}{\mathrm{Fr}^2}\rho\vec{e}_3 = 0,$$

$$\rho_t + (\mathbf{u} \cdot \nabla)\rho + \frac{\Lambda}{\mathrm{RoFr}^2}\mathbf{u} \cdot \vec{e}_2 - \Delta\rho = 0,$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$(x, y, z) \in \Omega = (0, L_x) \times (0, L_y) \times (0, 1).$$

The deviations $(\mathbf{u}, p, \rho)$ are assumed to be periodic in the x and y.
In the z direction we consider no-slip boundary conditions:

$$\mathbf{u} = 0, \quad \rho = 0, \qquad \text{at } z = 0, 1.$$

Functional Setting

$$X_1 = \{(\mathbf{u}, \rho) \in H^2(\Omega, \mathbb{R}^4) : \nabla \cdot \mathbf{u} = 0,\ \mathbf{u}\ |_{z=0,1} = 0,\ \rho\ |_{z=0,1} = 0\}$$
$$X = \{(\mathbf{u}, \rho) \in L^2(\Omega, \mathbb{R}^4) : \nabla \cdot \mathbf{u} = 0,\ \mathbf{u}\ |_{z=0,1} = 0,\ \rho\ |_{z=0,1} = 0\}$$

where the spaces denote the usual Sobolev spaces such that $\mathbf{u}$, ρ are periodic in x and y with periods L_x and L_y.

$$\frac{d\phi}{dt} = L\phi + G(\phi), \qquad \phi = (\mathbf{u}, \rho), \tag{1}$$

where $L : X_1 \to X$ is the linear operator and $G : X_1 \times X_1 \to X$ is the nonlinear operator

$$\begin{aligned} L\phi &= -\mathcal{P} \begin{pmatrix} \Lambda z \frac{\partial \mathbf{u}}{\partial x} + \Lambda w \hat{e}_3 - \mathrm{Pr}\Delta \mathbf{u} + \frac{1}{\mathrm{Ro}} \hat{e}_3 \times \mathbf{u} + \frac{1}{\mathrm{Fr}^2} \rho \hat{e}_3 \\ \frac{\Lambda}{\mathrm{RoFr}^2} v - \Delta \rho \end{pmatrix}, \\ G(\phi, \tilde{\phi}) &= -\mathcal{P} \begin{pmatrix} (\mathbf{u} \cdot \nabla)\tilde{\mathbf{u}} \\ (\mathbf{u} \cdot \nabla)\tilde{\rho} \end{pmatrix}, \\ G(\phi) &= G(\phi, \phi). \end{aligned} \tag{2}$$

Here $\mathcal{P} : L^2(\Omega, \mathbb{R}^4) \to X$ is the standard Leray projection.

Energy Stability Theorem

If

$$\Lambda < \mathrm{Fr}^4\,\mathrm{Ro}\,\mathrm{Pr}\,\pi^4,$$

then the energy

$$E = \|\mathbf{u}\|^2 + \frac{\mathrm{Ro}}{\Lambda}\,\|\rho\|^2$$

decays in the L^2 norm $\|\cdot\|$ at least exponentially in time.

Increasing Λ is destabilizing, increasing Fr, Ro and Pr are stabilizing.

Proof.

Taking L^2 inner product between the main equations and $(\mathbf{u}, \rho)$,

$$\frac{dE}{2dt} = -\Lambda\,\|w\|^2 - \mathrm{Pr}\,\|\nabla\mathbf{u}\|^2 - \frac{1}{\mathrm{Fr}^2}\int_\Omega \rho(w+v)dV - \frac{\mathrm{Ro}}{\Lambda}\,\|\nabla\rho\|^2 .$$

Using Poincare inequalities (thanks to the no-slip boundary conditions), Cauchy-Schwarz and the Young inequalities, dropping some terms, we arrive at

$$\frac{dE}{2dt} \le \left(-\mathrm{Pr}\pi^2 + \frac{\Lambda}{\mathrm{Fr}^4\mathrm{Ro}\pi^2}\right)\|\mathbf{u}\|^2 - \frac{\mathrm{Ro}\pi^2}{2\Lambda}\,\|\rho\|^2 .$$

By the energy stability condition, $\mathrm{Pr}\pi^2 - \frac{\Lambda}{\mathrm{Fr}^4\mathrm{Ro}\pi^2} > 0$.

$$\frac{dE}{dt} \le -c_0 E(t),$$

□

Principle of Exchange of Stabilities

Consider the eigenvalue problem for the linear operator.

$$L\phi = \beta\phi$$

Denote the eigenvalues $\beta = \beta_{m_x,m_y,m_z}$ and eigenvectors

$$\phi = \phi_{m_x,m_y,m_z} = \begin{bmatrix} \mathbf{u} \\ \rho \end{bmatrix} = e^{i\left(\frac{m_x\pi x}{L_x} + \frac{m_y\pi y}{L_y}\right)} \tilde{\phi}_{m_x,m_y,m_z}(z)$$

Let $\mathrm{Re}\,\beta_{m_x^c,m_y^c,1}(\lambda)$ be the largest among all $\mathrm{Re}\,\beta_{m_x,m_y,m_z}(\lambda)$.
Assume the existence of a critical parameter

$$\Lambda_c = \Lambda_c(L_x, L_y, \mathrm{Fr}, \mathrm{Pr}, \mathrm{Ro})$$

$$\mathrm{Re}\,\beta_{m_x^c,m_y^c,1} = \mathrm{Re}\,\beta_{m_x^c,m_y^c,2} \begin{cases} < 0 & \text{if } \Lambda < \Lambda_c, \\ = 0 & \text{if } \Lambda = \Lambda_c, \\ > 0 & \text{if } \Lambda > \Lambda_c, \end{cases}$$

$$\mathrm{Re}\,\beta_{m_x,m_y,m_z}(\Lambda_c) < 0 \qquad \text{for } (m_x, m_y) \neq (m_x^c, m_y^c) \text{ or } m_z \notin \{1, 2\}.$$

We can only show that the above condition is numerically verified.

Dynamic Transition Theory I

$$\frac{d\phi}{dt} = L_\lambda \phi + G(\phi, \lambda), \quad \phi(0) = \phi_0.$$

$\phi = 0$ is an equilibrium solution for any $\lambda = \lambda(L_x, L_y, \Lambda, \mathrm{Fr}, \mathrm{Pr}, \mathrm{Ro})$.

Definition

We say that the system undergoes a dynamic transition from $\phi = 0$ at $\Lambda = \Lambda_c$ if

1. $\phi = 0$ is locally asymptotically stable for $\Lambda < \Lambda_c$,
2. there exists a neighborhood $U \subset H$ of $\phi = 0$ independent of λ, such that for any $\phi_0 \in U \setminus \Gamma_\lambda$, the solution $\phi(t, \phi_0, \lambda)$ satisfies

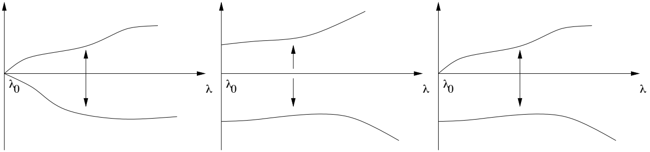
$$\limsup_{t\to\infty} \|u_\lambda(t, u_0)\| \geq \delta(\lambda) > 0, \quad \lim_{\lambda\to\lambda_0} \delta(\lambda) \geq 0.$$

where Γ_λ is the stable manifold of $\phi = 0$ with codimension ≥ 1.

DYNAMIC TRANSITION THEORY II

Ma Wang (2013) Classification of Dynamic Transitions of Dissipative Systems

The transition states are represented by a local attractor.



(a) Continuous Transition.

(b) Jump Transition. Drastic change.

(c) Mixed Transition. Randomness depending on fluctuations.

Transition Theorem

Let $\beta_c = \beta_{m_x^c, m_y^c, 1}$ and $\phi_c = \phi_{m_x^c, m_y^c, 1}$. There exists a number $A \in \mathbb{C}$.

1. If $\mathrm{Im}(\beta_c) = 0$ then $A \in \mathbb{R}$.
   1. If $A < 0$, then the system undergoes a continuous dynamic transition on $\Lambda > \Lambda_c$ and bifurcates, on $\Lambda > \Lambda_c$, from the basic shear flow to an attracting circle of steady states:
   $$\Sigma_\Lambda = \left\{ 2\sqrt{\frac{-\beta_c}{A}}\,\mathrm{Re}(e^{i\gamma}\phi_c) + \mathrm{O}(|\beta_c|) : \gamma \in \mathbb{R} \right\}.$$
   2. If $A > 0$, then the system undergoes jump transition, and bifurcates, on $\Lambda < \Lambda_c$, from the basic shear flow to a repeller Σ_Λ having the same form as given above.
2. If $\mathrm{Im}(\beta_c) \neq 0$ then $A \in \mathbb{C} \setminus \mathbb{R}$.
   1. If $\mathrm{Re}(A) < 0$, then the system undergoes a continuous transition, and bifurcates, on $\Lambda > \Lambda_c$, from the basic shear flow to a stable limit cycle given by
   $$\phi_{\text{bif}} = z(t)\phi_c + \overline{z(t)\phi_c} + O(-\mathrm{Re}(\beta_c)),$$
   where
   $$z(t) = \sqrt{\frac{-\mathrm{Re}(\beta_c)}{\mathrm{Re}(A)}}\exp\left(i\,\mathrm{Im}(\beta_c)t\right).$$
   2. If $\mathrm{Re}(A) > 0$, there is a jump transition and bifurcation on $\Lambda < \Lambda_c$, from the basic shear flow to an unstable periodic solution.

Transition Number

$$A = \sum_{m_z=1}^{\infty} A_{0,m_z} + A_{2,m_z} \in \mathbb{C}$$

The numbers $A_{0,j}$ and $A_{2,j}$ represent the nonlinear interactions of the critical modes with the modes having wave numbers 0 and $2\alpha_c$ respectively.

$$\begin{aligned}
A_{0,m_z} &= \frac{1}{\langle \phi_c, \phi_c^* \rangle} \Phi_{0,m_z} \langle G_s(\phi_c, \phi_{0,m_z}), \phi_c^* \rangle, \\
A_{2,m_z} &= \frac{1}{\langle \phi_c, \phi_c^* \rangle} \Phi_{2c,m_z} \langle G_s(\overline{\phi_c}, \phi_{2c,m_z}), \phi_c^* \rangle, \\
\Phi_{0,m_z} &= \frac{1}{-\beta_{0,m_z} \langle \phi_{0,m_z}, \phi_{0,m_z}^* \rangle} \langle G_s(\phi_c, \overline{\phi_c}), \phi_{0,m_z}^* \rangle, \\
\Phi_{2c,m_z} &= \frac{1}{(2i\,\mathrm{Im}(\beta_c) - \beta_{2c,m_z}) \langle \phi_{2c,m_z}, \phi_{2c,m_z}^* \rangle} \langle G(\phi_c, \phi_c), \phi_{2c,m_z}^* \rangle.
\end{aligned}$$

$$\begin{aligned}
&\phi_c = \phi_{m_x^c, m_y^c, 1}, \quad \phi_{0,m_z} = \phi_{0,0,m_z}, \quad \phi_{2c,m_z} = \phi_{2m_x^c, 2m_y^c, m_z}, \\
&\phi_c^* = \phi_{m_x^c, m_y^c, 1}^*, \quad \phi_{0,m_z}^* = \phi_{0,0,m_z}^*, \quad \phi_{2c,m_z}^* = \phi_{2m_x^c, 2m_y^c, m_z}^*, \\
&\beta_c = \beta_{m_x^c, m_y^c, 1}, \quad \beta_{0,m_z} = \beta_{0,0,m_z}, \quad \beta_{2c,m_z} = \beta_{2m_x^c, 2m_y^c, m_z}.
\end{aligned}$$

Numerical Solution of the Eigenvalue Problem

For eigenvalues with $\alpha \neq 0$, the eigenvalue problem becomes

$$
\begin{aligned}
&\Pr(D^2-\alpha^2)C - \Lambda i\alpha_x zC - \frac{1}{\mathrm{Ro}}DW = \beta C,\\
&\frac{1}{\mathrm{Ro}}DC + \Pr(D^2-\alpha^2)^2 W - \Lambda\alpha^2 W - \Lambda i\alpha_x\left(D(zDW) - \alpha^2 zW\right) + \frac{\alpha^2}{\mathrm{Fr}^2}R = \beta(D^2-\alpha^2)W,\\
&-\frac{\Lambda}{\mathrm{RoFr}^2\alpha^2}(i\alpha_x C + i\alpha_y DW) + (D^2-\alpha^2)R = \beta R,\\
&C = W = DW = R = 0 \qquad \text{at } z = 0, 1.
\end{aligned}
$$

Here C is the vertical component of the vorticity,

$$
\begin{aligned}
-i\alpha_x V + i\alpha_y U &= C,\\
i\alpha_x U + i\alpha_y V &= -DW.
\end{aligned}
$$

$z \to \tilde{z} = 2z - 1.$

$$
C = \sum_{i=0}^{N-1} \widehat{C}_i e_i^C(\tilde{z}), \qquad W = \sum_{i=0}^{N-1} \widehat{W}_i e_i^W(\tilde{z}), \qquad R = \sum_{i=0}^{N-1} \widehat{R}_i e_i^R(\tilde{z}). \tag{3}
$$

The basis functions carefully chosen combinations of Legendre polynomials.

$$
e_i^C(\tilde{z}) = e_i^W(\tilde{z}) = D^2 e_i^W(\tilde{z}) = e_i^R(\tilde{z}) = 0, \quad \tilde{z} = -1, 1.
$$

Phase Transition Diagrams

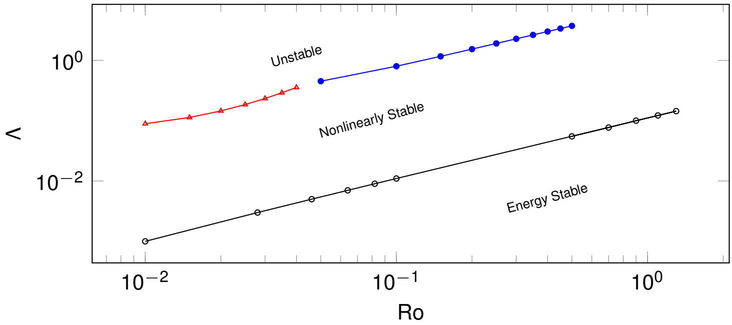


FIGURE: Stability curves for $\mathrm{Pr} = 0.71$, $\mathrm{Fr} = 0.2$, $L_x = L_y = 1.0$. The curve indicates transition to multiple equilibria (blue line with filled circles), transition to spatio-temporal oscillations (red line with triangles) and energy stability (black line with open circles).

More of these diagrams in the paper.

Transition States

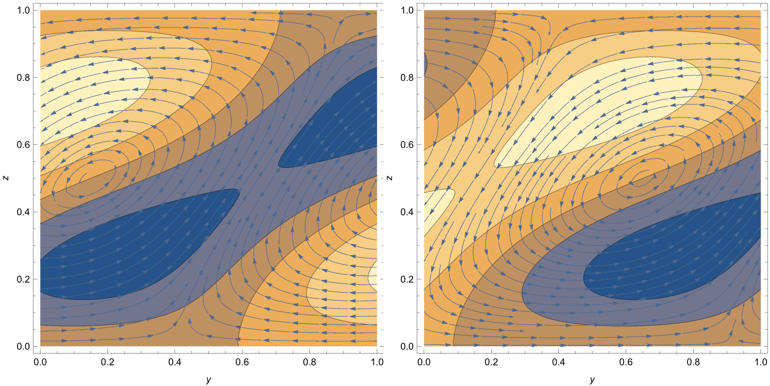


FIGURE: The real and imaginary parts of (v_c, w_c)-vector field on top of the contour plot of the critical density field in the yz-plane for $L_x = L_y = 1$, $\mathrm{Fr} = 0.2$, $\mathrm{Pr} = 0.71$, $\mathrm{Ro} = 0.1$, $\Lambda = \Lambda_c$. Here $(m_x^c, m_y^c) = (0, 1)$.

Proof of Main Theorem I

$$\begin{aligned} \frac{d\phi}{dt} &= L_\lambda \phi + G(\phi, \lambda), \\ \phi(0) &= \varphi, \end{aligned} \tag{4}$$

By the spectral theorem, the spaces X_1 and X can be decomposed into the direct sum

$$X_1 = E_1^\lambda \oplus E_2^\lambda, \quad X = E_1^\lambda \oplus \overline{E_2^\lambda},$$

where

$$\begin{aligned} E_1^\lambda &= \text{span}\{z\phi_c(\lambda) + \overline{z\phi_c(\lambda)} \mid z \in \mathbb{C}\}, \\ E_2^\lambda &= \text{the complement of } E_1^\lambda \text{ in } X. \end{aligned}$$

Then L_λ is invariant on E_1^λ and E_2^λ, i.e., L_λ can be decomposed as

$$\begin{aligned} L_\lambda &= \mathcal{J}_\lambda \oplus \mathcal{L}_\lambda, \\ \mathcal{J}_\lambda &: E_1^\lambda \to E_1^\lambda, \\ \mathcal{L}_\lambda &: X_1 \cap E_2^\lambda \to E_2^\lambda. \end{aligned} \tag{5}$$

Let P_i be the projection onto E_i^λ.

Proof of Main Theorem II

Theorem

Under the above assumptions, for $\lambda_0 \leq \lambda < \lambda_0 + \delta$ where δ is sufficiently small, there exists a local manifold $M = M(\lambda)$ called the center manifold such that

- *M is invariant.*
- *M is tangent to E_1.*
- *M is given by the graph of a center manifold function $\Phi : B \subset E_1 \to E_2$ where B is a neighborhood of $0 \in E_1$.*
- *If $u(t) = x(t) + y(t)$ is a solution such that $x \in E_1$ and $y \in E_2$, then there exists $k > 0$ and $\beta > 0$,*
$$\|y(t) - \Phi(x(t))\| \leq ke^{-\beta t}.$$

Proof of Main Theorem III

Under the above assumptions, for λ sufficiently close to λ_0 we prove the following approximation for the center manifold function.

$$\Phi(x(t), \lambda) = (2i\,\mathrm{Im}(\beta_1) - \mathcal{L})^{-1} P_2 G(z\phi_1, z\phi_1) + (-\mathcal{L})^{-1} P_2 G(z\phi_1, \overline{z\phi_1}) + \mathrm{o}(2) + \text{ c.c.}, \tag{6}$$

where c.c. stands for the complex conjugate of the whole expression coming before and $\mathrm{o}(k)$ stands for

$$\mathrm{o}(k) := \mathrm{o}(\|x\|_{X_1}^k) + \mathrm{O}(|\mathrm{Re}\,\beta_1(\lambda)|\,\|x\|_{X_1}^k). \tag{7}$$

Using this approximation, we carry out reduction on to the center manifold and study the dynamics of the reduced system.

Thank You

Thank You!

Questions please.