Disconjugacy via Lyapunov and Vallée-Poussin type inequalities for forced differential equations

Abdullah Özbekler [a]
(joint work with Ravi P. Agarwal [b])

[a] *Atılım University, Department of Mathematics, Ankara-Turkey*
[b] *Texas A&M University-Kingsville, Department of Mathematics, Texas-USA*

8th WDEA2017, June 2-4, 2017, Dokuz Eylül University, İzmir-Turkey

1. Introduction

2. Literature

3. Motivation

4. Main Results

5. Examples

6. References

1. Introduction

$$(r(t)|x'|^{\beta-1}x')' + q(t)|x|^{\gamma-1}x = f(t); \tag{1}$$

Dirichlet Boundary Conditions:

$$x(a) = x(b) = 0 \tag{BC}$$

(i) $\gamma \in (0, 2\beta)$

(ii) r, q and f are real-valued functions with $r(t) > 0$

(iii) a and b are consecutive zeros ($a < b$)

$(\beta = 1)$ $$(r(t)x')' + q(t)|x|^{\gamma-1}x = 0; \qquad \gamma > 0 \tag{2}$$

$\gamma = 1:$ **Linear**

$0 < \gamma < 1:$ **Sub-Linear** (Emden-Fowler)

$1 < \gamma:$ **Super-Linear** (Emden-Fowler)

$$(r(t)|x'|^{\beta-1}x')' + q(t)|x|^{\gamma-1}x = 0; \qquad \beta, \gamma > 0 \tag{3}$$

$\beta = \gamma: \ (\neq 1)$ **Half-Linear**

$\beta > \gamma:$ **Sub-Half-Linear**

$\beta < \gamma:$ **Super-Half-Linear**

1. C.V. Coffman, James S.W. Wong, 1972

2. R.A. Moore, Z. Nehari, 1959

3. G.B. Gustafson, K. Schmitt, 1972

4. Z. Nehari, 1960

5. R.M. Moroney, 1962

6. L.H. Erbe, 1970

7. James S. W. Wong, 1975

1. James S.W. Wong, M.K. Kwong, Y.G. Sun, (Linear, Emden-Fowler Sub/Super linear)

2. O. Dosly, P. Rehak, (J. Manojlovic), (Half-Linear, 2005)

3. J. Jaros, T. Kusano, N. Yoshida, (Super (Half) linear)

4. R.P. Agarwal, S.R. Grace, D. O'Regan, Half-Linear, Super-linear and Sublinear Dynamic Equations, 2002

2. Literature:

$$x''(t) + q(t)x(t) = 0 \quad (4)$$

$$x(a) = x(b) = 0 \quad (5)$$

Lyapunov's Ineq. (1907, A.M. Lyapunov) (R: 1947)

$$\int_a^b |q(t)| \mathrm{d}t > \frac{4}{b-a} \quad (6)$$

Lyapunov's Ineq. (1951, Wintner)

$$\int_a^b q^+(t) \mathrm{d}t > \frac{4}{b-a} \quad (7)$$

$$q^+(t) := \max\{q(t), 0\}$$

Hartman's Ineq. (1965, Hartman)

$$\int_a^b (b-t)(t-a)q^+(t)\mathrm{d}t > b-a \tag{8}$$

$$\left(\frac{b-a}{2} = \frac{(b-t)+(t-a)}{2} \geq \sqrt{(b-t)(t-a)}\right)$$

Selfadjoint Eq.

$$(r(t)x'(t))' + q(t)x(t) = 0 \tag{9}$$

$$\int_a^b q^+(t)\mathrm{d}t > 4\left(\int_a^b r^{-1}(t)\mathrm{d}t\right)^{-1} \tag{10}$$

Half-Linear Eq. $\gamma > 0$

$$(r(t)|x'|^{\gamma-1}x')' + q(t)|x|^{\gamma-1}x = 0, \qquad \textbf{(HL)}$$

Lyapunov Type Ineq. (2003, Yang)

$$\int_a^b q^+(t)\mathrm{d}t > 2^{\gamma+1}\left(\int_a^b r^{-1/\gamma}(t)\mathrm{d}t\right)^{-\gamma} \qquad (11)$$

Hartman Type Ineq. (2010, Sim and Lee) $r(t) = 1,\ q(t) > 0$

$$\int_a^b (b-t)^\gamma (t-a)^\gamma q(t)\mathrm{d}t > 2^{1-\gamma}(b-a)^\gamma \qquad (12)$$

Lyapunov Type Ineq. (2011, Wang)

$$\int_a^b \left(\int_a^t r^{-1/\gamma}(s)\mathrm{d}s\right)^\gamma \left(\int_t^b r^{-1/\gamma}(s)\mathrm{d}s\right)^\gamma q^+(t)\mathrm{d}t > C_\gamma \left(\int_a^b r^{-1/\gamma}(t)\mathrm{d}t\right)^\gamma$$

$$C_\gamma = \begin{cases} 1, & 0 < \gamma \le 1; \\ 2^{1-\gamma}, & \gamma > 1. \end{cases}$$

Lyapunov Type Ineq. (2012, Tang and He)

$$\int_a^b \left[\left(\int_a^t r^{-1/\gamma}(s)\mathrm{d}s\right)^{-\gamma} + \left(\int_t^b r^{-1/\gamma}(s)\mathrm{d}s\right)^{-\gamma}\right]^{-1} q^+(t)\mathrm{d}t > 1$$

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0 \tag{13}$$

$$x(a) = x(b) = 0 \tag{14}$$

Vallée-Poussin Ineq. (1929, C.D. Vallée Poussin)

$$\int_0^\infty \frac{\mathrm{d}t}{t^2 + p_0 t + q_0} \le \frac{b-a}{2} \tag{15}$$

$$p_0 = \max_{t \in [a,b]} |p(t)| \quad \text{and} \quad q_0 = \max_{t \in [a,b]} |q(t)|. \tag{16}$$

$$(|x'|^{\alpha-1}x')' + p(t)|x'|^{\alpha-1}x' + q(t)|x|^{\alpha-1}x = 0 \tag{17}$$

$$x(a) = x(b) = 0, \qquad \alpha > 0 \tag{18}$$

Vallée-Poussin Ineq. (1999, Dosly and Lomtatidze)

$$\int_0^\infty \frac{\mathrm{d}t}{\alpha t^{1+1/\alpha} + p_0 t + q_0} \le \frac{b-a}{2} \tag{19}$$

$$p_0 = \max_{t\in[a,b]} |p(t)| \quad \text{and} \quad q_0 = \max_{t\in[a,b]} |q(t)|. \tag{20}$$

Application Areas.

- Oscillation and Sturmian Theory
- Asymptotic Theory
- Conjugacy/Disconjugacy
- Eigenvalue Problems
- Boundary Value Problems

Conjugacy/Disconjugacy.

$$x''(t) + q(t)x(t) = 0 \tag{21}$$

$$x(a) = x(b) = 0 \tag{22}$$

a **and** b **are consequtive zeros :**

$$\underbrace{\int_a^b q^+(t)\mathrm{d}t > \frac{4}{b-a}}_{\text{(Lyapunov's Ineq.)}} \tag{23}$$

Disconjugacy Criteria :

$$\int_a^b q^+(t)\mathrm{d}t \leq \frac{4}{b-a} \tag{24}$$

3. Motivation

(i) Forced Emden-Fowler Eq.

$$(r(t)x')' + q(t)|x|^{\alpha-1}x = \underbrace{f(t)}_{\text{or}(=0)} \tag{25}$$

Lyapunov Type Ineq. ?

$\alpha > 1$ (Super-Linear) **?**

$\alpha = 1$ (Linear) **?**

$\alpha < 1$ (Sub-Linear) **?**

(ii) Forced Nonlinear Eq.'s

$$(r(t)|x'|^{\beta-1}x')' + q(t)|x|^{\gamma-1}x = \underbrace{f(t)}_{\text{or}(=0)} \tag{26}$$

Lyapunov Type Ineq. ?

$\gamma > \beta$ (Super-Half-Linear) **?**

$\gamma = \beta$ (Half-Linear) **?**

$\gamma < \beta$ (Sub-Half-Linear) **?**

(iii) Forced Nonlinear Eq.'s

$$(r(t)|x'|^{\beta-1}x')' + p(t)|x'|^{\beta-1}x' + q(t)|x|^{\gamma-1}x = f(t) \tag{27}$$

$\gamma = \beta = 1$ and $f(t) = 0$ (Linear) **OK**

$\gamma = \beta$ and $f(t) = 0$ (Half-Linear) **OK**

Vallée Poussin Type Ineq. ? When $f(t) \underbrace{=}_{\neq} 0$

$\gamma > \beta$ (Super-Half-Linear) **?** $\gamma = \beta$ (Half-Linear) **?** $\gamma < \beta$ (Sub-Half-Linear) **?**

$\gamma > \beta = 1$ (Super-Linear) **?** $\gamma = \beta = 1$ (Linear) **?** $\gamma < \beta = 1$ (Sub-Linear) **?**

(iv) Lemma.

$$Az^{2\beta} - Bz^{\gamma} + \Gamma_{\gamma\beta} A^{-\gamma/(2-\gamma)} B^{2\beta/(2-\gamma)} \geq 0 \tag{28}$$

for any $\gamma \in (0, 2\beta)$, where

$$\Gamma_{\gamma\beta} = (2\beta - \gamma)\gamma^{\gamma/(2\beta-\gamma)} \beta^{-2\tau/(2\beta-\gamma)} 2^{-2\beta/(2\beta-\gamma)}. \tag{29}$$

$$\begin{aligned} B = z = 0 \quad &\Longleftrightarrow \quad (=) \\ B > 0 \quad &\Longrightarrow \quad (>) \end{aligned} \tag{30}$$

$A{=}B{=}1$:

$$z^{\gamma} < z^{2\beta} + \Gamma_{\gamma\beta} \tag{31}$$

4. Main Results

1. Lyapunov Type Ineq. Suppose that a and b are consecutive zeros of a nontrivial solution of Eq.

$$(r(t)|x'|^{\beta-1}x')' + q(t)|x|^{\gamma-1}x = f(t). \tag{32}$$

If $x(t) > 0$ on (a,b), then the Ineq.

$$2\Gamma_{\gamma\beta}\int_a^b q^+(t)\mathrm{d}t + \int_a^b f^-(t)\mathrm{d}t > 2^{\beta+1}\sqrt{\Gamma_{\gamma\beta}}\left(\int_a^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} \tag{33}$$

holds, where $\gamma \in (0, 2\beta)$ and

$$f^-(t) = \max\{-f(t), 0\}$$

Proof. Let $c \in (a,b)$ be the least point of the local maxima of $x(t)$ in (a,b), i.e., $x'(c)=0$ and $x'(t)>0$ on $[a,c)$.

Hölder's Ineq. $\Longrightarrow$

$$
\begin{aligned}
x^{\beta+1}(c) &= \left(\int_a^c x'(t)\mathrm{d}t\right)^{\beta+1} \\
&= \left(\int_a^c r^{-1/(\beta+1)}(t) r^{1/(\beta+1)}(t) x'(t)\mathrm{d}t\right)^{\beta+1} \\
&\le \left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t\right)^{\beta} \int_a^c r(t)[x'(t)]^{\beta+1}\mathrm{d}t. \qquad (34)
\end{aligned}
$$

$$(r(t)|x'|^{\beta-1}x')' + q(t)|x|^{\gamma-1}x = f(t) \qquad (\times x(t)) \tag{35}$$

Int. a to c :

$$\int_a^c r(t)[x'(t)]^{\beta+1}\mathrm{d}t = \int_a^c q(t)x^{\gamma+1}(t)\mathrm{d}t - \int_a^c f(t)x(t)\mathrm{d}t$$

$$\leq x^{\gamma+1}(c)\int_a^c q^+(t)\mathrm{d}t + x(c)\int_a^c f^-(t)\mathrm{d}t \tag{36}$$

$$\Longrightarrow \qquad x^{\beta}(c) \leq \mathscr{K}_0 x^{\gamma}(c) + \mathscr{K}_1 \tag{37}$$

$$\mathscr{K}_0 = \left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t\right)^{\beta}\int_a^c q^+(t)\mathrm{d}t \tag{38}$$

$$\mathscr{K}_1 = \left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t\right)^{\beta}\int_a^c f^-(t)\mathrm{d}t \tag{39}$$

Lemma $\Longrightarrow$ $x^{\gamma}(c) < x^{2\beta}(c) + \Gamma_{\gamma\beta}$

$\Longrightarrow$

$$\mathcal{K}_0 x^{2\beta}(c) - x^{\beta}(c) + \mathcal{K}_0\Gamma_{\gamma\beta} + \mathcal{K}_1 > 0 \tag{40}$$

$\Longrightarrow$

$$\mathcal{K}_0(\Gamma_{\gamma\beta}\mathcal{K}_0 + \mathcal{K}_1) > 1/4 \tag{41}$$

i.e.

$$\begin{aligned} \frac{1}{4}\left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t\right)^{-2\beta} &< \Gamma_{\gamma\beta}\left(\int_a^c q^{+}(t)\mathrm{d}t\right)^2 \\ &+ \left(\int_a^c q^{+}(t)\mathrm{d}t\right)\left(\int_a^c f^{-}(t)\mathrm{d}t\right). \end{aligned} \tag{42}$$

Mult. both side by $4\Gamma_{\gamma\beta}$ and completing square :

$$\sqrt{\Gamma_{\gamma\beta}}\left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} < 2\Gamma_{\gamma\beta}\int_a^c q^+(t)\mathrm{d}t + \int_a^c f^-(t)\mathrm{d}t. \tag{43}$$

Similarly, if $d \in (a,b)$ be the greatest point of the local maxima of $x(t)$ in (a,b), i.e., $x'(d) = 0$ and $x'(t) < 0$ on $(d,b]$. $\Longrightarrow$

$$\sqrt{\Gamma_{\gamma\beta}}\left(\int_d^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} < 2\Gamma_{\gamma\beta}\int_d^b q^+(t)\mathrm{d}t + \int_d^b f^-(t)\mathrm{d}t. \tag{44}$$

Adding;

$$\sqrt{\Gamma_{\gamma\beta}}\left\{\left(\int_a^c r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} + \left(\int_d^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta}\right\}$$
$$< 2\Gamma_{\gamma\beta}\int_a^b q^+(t)\mathrm{d}t + \int_a^b f^-(t)\mathrm{d}t. \tag{45}$$

Jensen's Ineq. $c \in [0,1],\ u_{1,2} \geq 0,\ \beta > 0$

$$\Longrightarrow \qquad (cu_1 + (1-c)u_2)^{-\beta} \leq cu_1^{-\beta} + (1-c)u_2^{-\beta}$$

$$c = 1/2 \Longrightarrow u_1^{-\beta} + u_2^{-\beta} \geq 2^{\beta+1}(u_1 + u_2)^{-\beta}$$

$$\Bigg(\underbrace{\int_a^c r^{-1/\beta}(t)\mathrm{d}t}_{u_1}\Bigg)^{-\beta} + \Bigg(\underbrace{\int_d^b r^{-1/\beta}(t)\mathrm{d}t}_{u_2}\Bigg)^{-\beta}$$

$$\geq 2^{\beta+1}\Bigg(\underbrace{\int_a^c r^{-1/\beta}(t)\mathrm{d}t}_{u_1} + \underbrace{\int_d^b r^{-1/\beta}(t)\mathrm{d}t}_{u_2}\Bigg)^{-\beta}$$

$$\geq 2^{\beta+1}\left(\int_a^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta}. \qquad (46)$$

2. Lyapunov Type Ineq. Suppose that a and b are consecutive zeros of a nontrivial solution of Eq.

$$(r(t)|x'|^{\beta-1}x')' + q(t)|x|^{\gamma-1}x = f(t). \tag{47}$$

If $x(t) < 0$ on (a,b), then the Ineq.

$$2\Gamma_{\gamma\beta}\int_a^b q^+(t)\mathrm{d}t + \int_a^b f^+(t)\mathrm{d}t > 2^{\beta+1}\sqrt{\Gamma_{\gamma\beta}}\left(\int_a^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} \tag{48}$$

holds, where $\gamma \in (0, 2\beta)$.

Proof. In fact, if $x(t) < 0$ for $t \in (a,b)$, then $z(t) := -x(t) > 0$ solution of

$$(r(t)|z'|^{\beta-1}z')' + q(t)|z|^{\gamma-1}z = -f(t) \tag{49}$$

3. Lyapunov Type Ineq. Suppose that a and b are consecutive zeros of a nontrivial solution of Eq.

$$(r(t)|x'|^{\beta-1}x')' + q(t)|x|^{\gamma-1}x = f(t). \tag{50}$$

If $x(t) \neq 0$ on (a,b), then the Ineq.

$$2\Gamma_{\gamma\beta}\int_a^b q^+(t)\mathrm{d}t + \int_a^b |f(t)|\mathrm{d}t > 2^{\beta+1}\sqrt{\Gamma_{\gamma\beta}}\left(\int_a^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} \tag{51}$$

holds, where $\gamma \in (0, 2\beta)$.

Proof. $|f(t)| \geq f^{\pm}(t)$ for all t.

Concluding Results/Remarks

$$(r(t)|x'|^{\beta-1}x')' + q(t)|x|^{\gamma-1}x = f(t) \tag{52}$$

A. (Forced) Sub-Half-Linear Eq.'s: $\gamma \in (0, \beta)$

$$2\Gamma_{\gamma\beta}\int_a^b q^+(t)\mathrm{d}t + \int_a^b |f(t)|\mathrm{d}t > 2^{\beta+1}\sqrt{\Gamma_{\gamma\beta}}\left(\int_a^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} \tag{53}$$

B. (Forced) Super-Half-Linear Eq.'s: $\gamma \in (\beta, 2\beta)$

$$2\Gamma_{\gamma\beta}\int_a^b q^+(t)\mathrm{d}t + \int_a^b |f(t)|\mathrm{d}t > 2^{\beta+1}\sqrt{\Gamma_{\gamma\beta}}\left(\int_a^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} \tag{54}$$

B^1. (Forced) Super-Half-Linear Eq.'s: $\gamma \in [2\beta, \infty)$

Open Problem!

C. (Forced) Half-Linear Eq.'s: $\gamma = \beta$ $(\Gamma_{\beta\beta} = 1/4)$

$$\int_a^b q^+(t)\mathrm{d}t + 2\int_a^b |f(t)|\mathrm{d}t > 2^{\beta+1}\left(\int_a^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} \qquad (55)$$

C^1. Half-Linear Eq.'s: $\gamma = \beta$ $(f(t) = 0)$

Existing Result:

$$\int_a^b q^+(t)\mathrm{d}t > 2^{\beta+1}\left(\int_a^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} \qquad (56)$$

$\beta = \mathbf{1}:$ $$(r(t)x')' + q(t)|x|^{\gamma-1}x = f(t) \tag{57}$$

D. (Forced) Emden-Fowler Sub-Linear: $\gamma \in (0, 1)$

$$2\Gamma_{\gamma 1}\int_a^b q^+(t)\mathrm{d}t + \int_a^b |f(t)|\mathrm{d}t > 4\sqrt{\Gamma_{\gamma 1}}\left(\int_a^b r^{-1}(t)\mathrm{d}t\right)^{-1} \tag{58}$$

E. (Forced) Emden-Fowler Super-Linear: $\gamma \in (1, 2)$

$$2\Gamma_{\gamma 1}\int_a^b q^+(t)\mathrm{d}t + \int_a^b |f(t)|\mathrm{d}t > 4\sqrt{\Gamma_{\gamma 1}}\left(\int_a^b r^{-1}(t)\mathrm{d}t\right)^{-1} \tag{59}$$

E^1. (Forced) Emden-Fowler Super-Linear: $\gamma \in [2, \infty)$

Open Problem!

F. (Forced) Linear Eq.'s: $\gamma = 1$ $(\Gamma_{11} = 1/4)$

$$\int_a^b q^+(t)\mathrm{d}t + 2\int_a^b |f(t)|\mathrm{d}t > 4\left(\int_a^b r^{-1}(t)\mathrm{d}t\right)^{-1} \tag{60}$$

F^1. Linear Eq.'s: $\gamma = 1$ $(f(t) = 0)$

Existing Result: $$\int_a^b q^+(t)\mathrm{d}t > 4\left(\int_a^b r^{-1}(t)\mathrm{d}t\right)^{-1} \tag{61}$$

(Forced) Mixed Non-Linear Eq.'s: $0 < \gamma < 1 < \beta < 2$

$$x'' + p(t)|x|^{\beta-1}x + q(t)|x|^{\gamma-1}x = f(t) \tag{62}$$

4. Hartman Type Ineq. Suppose that a, b, $a < b$, are consecutive zeros of a nontrivial solution of Eq. (62), then the Ineq.

$$\begin{aligned} &\left(\int_a^b (b-t)(t-a)(p^+ + q^+)(t)\mathrm{d}t \right) \\ &\quad \times \left(\int_a^b (b-t)(t-a)\left\{\beta_0 p^+(t) + \gamma_0 q^+(t) + |f(t)|\right\}\mathrm{d}t \right) \\ &> (b-a)^2/4 \end{aligned} \tag{63}$$

holds.

$$\beta_0 = (2-\beta)\beta^{\beta/(2-\beta)}2^{2/(\beta-2)} > 0 \tag{64}$$

$$\gamma_0 = (2-\gamma)\gamma^{\gamma/(2-\gamma)}2^{2/(\gamma-2)} > 0. \tag{65}$$

$(b-a)^2/4 \geq (b-t)(t-a) \quad \Longrightarrow$

5. Lyapunov Type Ineq.

$$\left(\int_a^b (p^+ + q^+)(t)\mathrm{d}t\right)\left(\int_a^b \left\{\beta_0 p^+(t) + \gamma_0 q^+(t) + |f(t)|\right\}\mathrm{d}t\right) > 4/(b-a)^2 \tag{66}$$

Concluding Results

$$x'' + p(t)|x|^{\beta-1}x + q(t)|x|^{\gamma-1}x = f(t) \tag{67}$$

a. (Forced) Emden-Fowler Sub-Linear: $p(t) = 0$

Hartman Type Ineq.

$$\left(\int_a^b (b-t)(t-a)q^+(t)\mathrm{d}t\right)\left(\int_a^b (b-t)(t-a)\left\{\gamma_0 q^+(t) + |f(t)|\right\}\mathrm{d}t\right) > (b-a)^2/4 \tag{68}$$

Lyapunov Type Ineq.

$$\left(\int_a^b q^+(t)\mathrm{d}t\right)\left(\int_a^b \{\gamma_0 q^+(t) + |f(t)|\}\mathrm{d}t\right) > 4/(b-a)^2 \tag{69}$$

b. (Forced) Emden-Fowler Super-Linear: $q(t) = 0$

Hartman Type Ineq.

$$\left(\int_a^b (b-t)(t-a)p^+(t)\mathrm{d}t\right)\left(\int_a^b (b-t)(t-a)\Big\{\beta_0 p^+(t) + |f(t)|\Big\}\mathrm{d}t\right) > (b-a)^2/4 \tag{70}$$

Lyapunov Type Ineq.

$$\left(\int_a^b p^+(t)\mathrm{d}t\right)\left(\int_a^b \{\beta_0 p^+(t) + |f(t)|\}\mathrm{d}t\right) > 4/(b-a)^2 \tag{71}$$

$$x'' + p(t)|x|^{\beta-1}x + q(t)|x|^{\gamma-1}x = f(t) \tag{72}$$

c. (Forced) Linear: $\beta \to 1^+$ and $\gamma \to 1^-$ $(\beta_0, \gamma_0 \to 1/4)$

$$x'' + [\underbrace{p(t) + q(t)}_{\nu(t)}]x = f(t) \tag{73}$$

Hartman Type Ineq.

$$\left(\int_a^b (b-t)(t-a)\nu^+(t)\mathrm{d}t \right) \left(\int_a^b (b-t)(t-a)\left\{ \nu^+(t) + 4|f(t)| \right\}\mathrm{d}t \right) > (b-a)^2 \tag{74}$$

Lyapunov Type Ineq.

$$\left(\int_a^b \nu^+(t)\mathrm{d}t \right) \left(\int_a^b \left\{ \nu^+(t) + 4|f(t)| \right\}\mathrm{d}t \right) > 16/(b-a)^2 \tag{75}$$

Forced Sub-Half-Linear Eq.'s with Damping. $0 < \gamma < \beta$

$$(r(t)|x'|^{\beta-1}x')' + p(t)|x'|^{\beta-1}x' + q(t)|x|^{\gamma-1}x = f(t) \tag{76}$$

6. Vallée Poussin Type Ineq. Suppose that a, b, $a < b$, are consecutive zeros of a nontrivial solution of Eq. (76), then the Ineq.

$$\int_0^\infty \frac{\mathrm{d}t}{\beta t^{1+1/\beta} + p_0 t + \lambda f_0^{1-\beta/\gamma} q_0^{\beta/\gamma}} \le \frac{b-a}{2} \tag{77}$$

holds.

$$p_0 = \max_{t\in[a,b]} |p(t)|, \quad q_0 = \max_{t\in[a,b]} |q(t)|, \quad f_0 = \min_{t\in[a,b]} |f(t)| > 0 \tag{78}$$

$$\lambda = \gamma\beta^{-\beta/\gamma}(\beta-\gamma)^{\beta/\gamma-1}.$$

Concluding Results/Remarks

$\beta \to \gamma$ and $f(t) = 0$:

$$(r(t)|x'|^{\gamma-1}x')' + p(t)|x'|^{\gamma-1}x' + q(t)|x|^{\gamma-1}x = 0 \qquad (79)$$

a. Half-Linear Eq.'s with Damping.: $\lambda \to 1$

Existing Result:
$$\int_0^\infty \frac{\mathrm{d}t}{\gamma t^{1+1/\gamma} + p_0 t + q_0} \leq \frac{b-a}{2} \qquad (80)$$

Convention : $0^0 = 1$

$\beta = 1 : 0 < \gamma < 1$

$$x'' + p(t)x' + q(t)|x|^{\gamma-1}x = f(t) \tag{81}$$

b. Forced Sub-Linear Eq.'s with Damping.:

Vallée Poussin Type Ineq.

$$\int_0^\infty \frac{\mathrm{d}t}{t^2 + p_0 t + \gamma(1-\gamma)^{1/\gamma-1} f_0{}^{1-1/\gamma} q_0{}^{1/\gamma}} \leq \frac{b-a}{2} \tag{82}$$

b^1**. Linear Eq.'s**: $\gamma \to 1^-$ $(f(t) = 0)$

Existing Result: $$\int_0^\infty \frac{\mathrm{d}t}{t^2 + p_0 t + q_0} \leq \frac{b-a}{2} \tag{83}$$

$$(r(t)|x'|^{\beta-1}x')' + p(t)|x'|^{\beta-1}x' + q(t)|x|^{\gamma-1}x = f(t) \qquad (84)$$

c. Vallée Poussin Type Ineq. Open Problems:

$\gamma = \beta = 1$ and $f(t) \neq 0$ (Linear) **?**

$\gamma = \beta$ and $f(t) \neq 0$ (Half-Linear) **?**

$\gamma < \beta$ and $f(t) = 0$ (Sub-Half-Linear) **?**

$\gamma < \beta = 1$ and $f(t) = 0$ (Sub-Linear) **?**

$\gamma > \beta$ (Super-Half-Linear) **?**

$\gamma > \beta = 1$ (Super-Linear) **?**

5. Examples

1.

$$x''(t) + 4\alpha^2 x(t) = \beta, \qquad |\alpha| + |\beta| \neq 0; \tag{85}$$

$$x(0) = x(b) = 0 \tag{86}$$

Lyapunov's Ineq. : $$b > \frac{\sqrt{2}}{\sqrt{2\alpha^2 + |\beta|}} \tag{87}$$

Disconjugacy on $[0, b]$: $$b \leq \frac{\sqrt{2}}{\sqrt{2\alpha^2 + |\beta|}} \tag{88}$$

$$x(0) = 0 \rightarrow \tag{89}$$

$$x(t) = c\sin(2\alpha t) + \frac{\beta}{4\alpha^2}\{1 - \cos(2\alpha t)\}, \qquad c \in \mathbb{R} \tag{90}$$

$$\text{Zeros:} \qquad t_n = n\pi/\alpha; \qquad n \in \mathbb{N} \tag{91}$$

$$\text{Disconjugate on } [0,b]: \qquad b \leq \frac{\sqrt{2}}{\sqrt{2\alpha^2 + |\beta|}} < \frac{\pi}{\alpha} \tag{92}$$

2.

$$((\mu + \cos t)x'(t))' + (2\cos t)x(t) = -\mu \sin t; \tag{93}$$

$$x(0) = x(\pi) = 0, \qquad \mu > 1 \tag{94}$$

$$\rightarrow \quad x(t) = \sin t \tag{95}$$

$x(0) = x(b) = 0:$

Lyapunov Ineq.:

$$\{1 + \mu(1 - \cos b)\} \arctan\left(\sqrt{\frac{\mu - 1}{\mu + 1}} \tan(b/2)\right) > \sqrt{\mu^2 - 1}/4 \tag{96}$$

$$\underbrace{\cdots \leq \cdots}$$

Disconjugacy on $[0,b]$

$b = \pi \rightarrow$ (96) OK

$\mu = 2$ $\to$ (Disconjugacy:)

$$H(b) := (3 - 2\cos b)\arctan\left(\frac{1}{\sqrt{3}}\tan(b/2)\right) \leq \sqrt{3}/4 \approx 0.433012$$

$$H(\pi/2) \approx 1.570796 \quad (\times)$$
$$H(\pi/3) \approx 0.643501 \quad (\times)$$
$$H(\pi/4) \approx 0.372242 \quad (\surd)$$
$$H(\pi/6) \approx 0.194602 \quad (\surd)$$

$\exists \gamma_0 \in (\pi/4, \pi/3)$ s.t $H(\gamma_0) = \sqrt{3}/4$ $\to$

Disconjugate on $[0, \gamma_0]$

6. REFERENCES:

1. A. M. Liapunov, *Probleme général de la stabilité du mouvement, (French Translation of a Russian paper dated 1893)*, Ann. Fac. Sci. Univ. Toulouse 2 (1907), 27247, Reprinted as Ann. Math. Studies, No. 17, Princeton, 1947.

2. A. Wintner, *On the nonexistence of conjugate points*, Amer. J. Math. **73** (1951), 368–380.

3. P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964 and Birkhuser, Boston 1982.

4. X. Yang, *On Liapunov-type inequality for certain higher-order differential equations*, Appl. Math. Comput. **134** (2003), 307–317.

5. Ch. De Vallée Poussin, *Sur l'équation différentielle linéqire du second order. Détermination d'une intéegrate par deux valuers assignés. Extension aux équasions d'ordre n*, J. Math. Pures Appl. **8** (1929), 125–144.

6. I. Sim, Y. H. Lee, *Lyapunov inequalities for one-dimensional* p*-Laplacian problems with a singular weight function*, J. Inequal. Appl. (2010), Art. ID 865096, 9 pp.

7. X. Wang, *Lyapunov type inequalities for second-order half-linear differential equations*, J. Math. Anal. Appl. **382** (2011), 792–801.

8. X. H. Tang, X. He, *Lower bouns for generalized eigenvalues of the quasilinear systems*, J. Math. Anal. Appl. **385** (2012), 72–85.

9. O. Došlý, A. Lomtatidze, *Disconjugacy and disfocality criteria for singular half-linear second order differential equations*, Ann. Polon. Math. **72** (1999), 273–284.

THANKS...