On the Numerical Inversion of a Generalized Radon Transform

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Integral Geometry Problems (IGP)

(In the sense that based on the work of Radon (Radon 1917))

- IGPs consist in determining a function by its given integrals of this function (or the integrals are taken with a certain weight multiplier) over a family of manifolds.

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- The classical Radon transform maps a function to its integrals over all hyperplanes in n-dimensional space.
- In general, the transforms involving integrations over curved surfaces and/or weight are called generalized Radon transforms.
- Applications: Imaging, Seismology, Astronomy, Radar, and many others fields.

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- Conditions to uniquely determine the unknown function,
- Derive stability estimates,
- Analytical formulas expressing the unknown function in terms of its given integrals,
- Numerical reconstruction algorithms
  - Filtered Backprojection (FBP),
  - Algebraic Reconstruction Techniques (ART), etc.

Outline

- Statement of the problem
- Relation between the GRT and the Radon transform (RT).
- Inversion formulas
- Numerical reconstruction

A Generalized Radon Transform (GRT)

Suppose that $f(x_1, x_2)$ is a continuous and compactly supported function on $\mathbb{R}^2$ and let φ be a real valued continuous function on $\mathbb{R}$.

- We study the problem of reconstructing the function f from the integrals

$$\mathcal{R}_\varphi f(s,u) = \int_{\mathbb{R}} f(x_1, u + s\varphi(x_1 - c))dx_1, \quad (1)$$

for $u, s \in \mathbb{R}$ and a fixed $c \in \mathbb{R}$.

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  for $u, s \in \mathbb{R}$ and a fixed $c \in \mathbb{R}$.

- $\mathcal{R}_\varphi f$ integrates f over the family of curves $x_2 = u + s\varphi(x_1 - c)$ with respect to the variable x_1,
- $\mathcal{R}_\varphi f$ can be considered as a generalized Radon transform (GRT).

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- Obtaining some back-projection type inversion formulas,
- Describe a method for the numerical reconstruction of f from its GRT,
- Demonstrating the feasibility of the proposed method by presenting some numerical results.

Assumptions; Family of Curves

The inversion of the GRT will be investigated by assuming that φ satisfies some smoothness and monotonicity conditions;

- **Case 1.** Let
  - φ be a smooth function on $\mathbb{R}$ and the derivative φ' be nonzero on $\mathbb{R}$,
  (Either $\varphi' > 0$ or $\varphi' < 0$ on $\mathbb{R}$)

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- **Case 1.** Let
  - φ be a smooth function on $\mathbb{R}$ and the derivative φ' be nonzero on $\mathbb{R}$,
  (Either $\varphi' > 0$ or $\varphi' < 0$ on $\mathbb{R}$)
- **Case 2.** Let
  - φ be a smooth function on $\mathbb{R}$ with $\varphi'(0) = 0$ or
  - φ be a smooth function on $\mathbb{R}\setminus\{0\}$ with $\varphi'(0)$ does not exist,
  (i.e. $x_1 = c$ be the critical point of $\varphi(x_1 - c)$,
  and the derivative φ' be nonzero on $\mathbb{R}\setminus\{0\}$.)
  (Either $\varphi'_{-} < 0$ or $\varphi'_{-} > 0$ on $\mathbb{R}^{-}$
  and either $\varphi'_{+} < 0$ or $\varphi'_{+} > 0$ on $\mathbb{R}^{+}$,
  where $\varphi|_{\mathbb{R}^{-}} = \varphi_{-}$ and $\varphi|_{\mathbb{R}^{+}} = \varphi_{+}$)

Classical Radon transform (RT)

- The inversion of the GRT will be investigated via the RT

$$\mathcal{R}k(\boldsymbol{\theta}, t) = \int_{\mathbb{R}} k(\boldsymbol{\theta} t + \tau \boldsymbol{\theta}^{\perp}) d\tau, \tag{2}$$

where $t \in \mathbb{R}$, $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$ and $\boldsymbol{\theta}^{\perp} = (-\sin\theta, \cos\theta)$

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- The integrals of k are taken over the all straight lines of the plane.

Lemma 1

To pose the relation between $\mathcal{R}_\varphi$ and $\mathcal{R}$, we define some functions k_φ on $\mathbb{R}^2$ as in the following Lemma.

Lemma

(i) For φ as in Case 1 we define

$$k_\varphi(y_1, y_2) = \begin{cases} f(\varphi^{-1}(y_1) + c, y_2)(\varphi^{-1})'(y_1) & , \text{ if } y_1 \in \text{range}(\varphi), \\ 0 & , \text{ otherwise}, \end{cases} \tag{3}$$

where φ^{-1} is the inverse of φ. Then

$$f(x_1, x_2) = k_\varphi(\varphi(x_1 - c), x_2)\varphi'(x_1 - c), \quad \text{on } \mathbb{R}^2. \tag{4}$$

Lemma 1

Lemma (continued)

(ii) For φ as in Case 2 we define

$$k_{\varphi_-}(y_1, y_2) = \begin{cases} f(\varphi_-^{-1}(y_1) + c, y_2)(\varphi_-^{-1})'(y_1) & , \text{ if } y_1 \in range\,(\varphi_-) \\ 0 & , \text{ otherwise} \end{cases} \quad (5)$$

$$k_{\varphi_+}(y_1, y_2) = \begin{cases} f(\varphi_+^{-1}(y_1) + c, y_2)(\varphi_+^{-1})'(y_1) & , \text{ if } y_1 \in range\,(\varphi_+) \\ 0 & , \text{ otherwise} \end{cases} \quad (6)$$

where φ_-^{-1} and φ_+^{-1} are the inverses of φ_- and φ_+, respectively. Then on $\mathbb{R} \setminus \{c\} \times \mathbb{R}$

$$f(x_1, x_2) = \begin{cases} k_{\varphi_-}(\varphi_-(x_1 - c), x_2)\varphi_-'(x_1 - c) & , \text{ if } x_1 < c, \\ k_{\varphi_+}(\varphi_+(x_1 - c), x_2)\varphi_+'(x_1 - c) & , \text{ if } x_1 > c. \end{cases} \quad (7)$$

Theorem 1 (The relation between GRT and RT)

Theorem

(i) Let $f \in C(\mathbb{R}^2)$ have compact support in $\mathbb{R}^2$. For φ as in Case 1, we have the relation

$$\sqrt{1+s^2}\mathcal{R}_\varphi f(s,u) = sgn(\varphi')\mathcal{R}k_\varphi\left(-\frac{s}{\sqrt{1+s^2}}, \frac{1}{\sqrt{1+s^2}}, \frac{u}{\sqrt{1+s^2}}\right). \tag{8}$$

(ii) Let $f \in C(\mathbb{R}^2)$ have compact support in $\mathbb{R}\backslash\{c\}\times\mathbb{R}$. For φ as in Case 2, we have the relation

$$\begin{aligned}\sqrt{1+s^2}\mathcal{R}_\varphi f(s,u) &= sgn(\varphi'_-)\mathcal{R}k_{\varphi_-}\left(-\frac{s}{\sqrt{1+s^2}}, \frac{1}{\sqrt{1+s^2}}, \frac{u}{\sqrt{1+s^2}}\right)\\ &+ sgn(\varphi'_+)\mathcal{R}k_{\varphi_+}\left(-\frac{s}{\sqrt{1+s^2}}, \frac{1}{\sqrt{1+s^2}}, \frac{u}{\sqrt{1+s^2}}\right)\end{aligned} \tag{9}$$

Analogue of the Fourier slice theorem

Theorem

(i) Let $f \in C^\infty(\mathbb{R}^2)$ have compact support in $\mathbb{R}^2$. Then, for φ as in Case 1, we have

$$\mathcal{F}_2\left(sgn(\varphi')k_\varphi\right)(\alpha,\beta) = \mathcal{F}_1\left(\mathcal{R}_\varphi f\right)\left(-\frac{\alpha}{\beta},\beta\right), \qquad (10)$$

(ii) Let $f \in C^\infty(\mathbb{R}^2)$ have compact support in $\mathbb{R}\setminus\{c\}\times\mathbb{R}$. Then, for φ as in Case 2, we have

$$\mathcal{F}_2\left(sgn(\varphi'_-)k_{\varphi_-} + sgn(\varphi'_+)k_{\varphi_+}\right)(\alpha,\beta) = \mathcal{F}_1\left(\mathcal{R}_\varphi f\right)\left(-\frac{\alpha}{\beta},\beta\right), \qquad (11)$$

where $\mathcal{F}_1$ is the $1-dimensional$ Fourier transform operator with respect to the second argument and $\mathcal{F}_2$ is the $2-dimensional$ Fourier transform operator.

Main theorem on the inversion

Theorem

(i) Let $f \in C^\infty(\mathbb{R}^2)$ have compact support in $\mathbb{R}^2$. Then, for φ as in Case 1 we have

$$f(x_1, x_2) = \frac{|\varphi'(x_1 - c)|}{2\pi} \int_{\mathbb{R}} \mathcal{H}\left(\partial_u \mathcal{R}_\varphi f\right)(s, x_2 - s\varphi(x_1 - c))\, ds, \quad (12)$$

(ii) Let $f \in C^\infty(\mathbb{R}^2)$ have compact support in $\mathbb{R}\setminus\{c\} \times \mathbb{R}$. Then, for φ as in Case 2, we have

$$f(x_1, x_2) = \begin{cases} \dfrac{|\varphi'_-(x_1 - c)|}{2\pi} \displaystyle\int_{\mathbb{R}} \mathcal{H}\left(\partial_u \mathcal{R}_\varphi f\right)(s, x_2 - s\varphi_-(x_1 - c))\, ds \\ -sgn(\varphi'_+) k_{\varphi_+}\left(\varphi_-(x_1 - c), x_2\right) \left|\varphi'_-(x_1 - c)\right|, \quad \text{if } x_1 < c, \\ \dfrac{|\varphi'_+(x_1 - c)|}{2\pi} \displaystyle\int_{\mathbb{R}} \mathcal{H}\left(\partial_u \mathcal{R}_\varphi f\right)(s, x_2 - s\varphi_+(x_1 - c))\, ds \\ -sgn(\varphi'_-) k_{\varphi_-}\left(\varphi_+(x_1 - c), x_2\right) \left|\varphi'_+(x_1 - c)\right|, \quad \text{if } x_1 > c, \end{cases} \quad (13)$$

where $\mathcal{H}$ is the Hilbert transform operator with respect to the second argument.

Some consequences I

φ'_- and φ'_+ have the same sign

If φ'_- and φ'_+ have the same sign, then $range\,(\varphi_-) \cap range\,(\varphi_+) = \emptyset$, and hence

$$\begin{aligned} sgn(\varphi'_+) k_{\varphi_+}\left(\varphi_-(x_1 - c), x_2\right) \left|\varphi'_-(x_1 - c)\right| &= 0, \\ sgn(\varphi'_-) k_{\varphi_-}\left(\varphi_+(x_1 - c), x_2\right) \left|\varphi'_+(x_1 - c)\right| &= 0, \end{aligned}$$

so we have

Corollary

Let $f \in C^\infty(\mathbb{R}^2)$ have compact support in $\mathbb{R} \setminus \{c\} \times \mathbb{R}$. Then, for φ as in Case 2 with φ'_- and φ'_+ have the same sign we have

$$f(x_1, x_2) = \frac{|\varphi'(x_1 - c)|}{2\pi} \int_{\mathbb{R}} \mathcal{H}\left(\partial_u \mathcal{R}_\varphi f\right)(s, x_2 - s\varphi(x_1 - c))\, ds.$$

Some consequences II

φ'_- **and** φ'_+ **have the opposite sign** (We consider the case for φ is even)

- If f is even with respect to the line $x_1 = c$, i.e. $f(-x_1 + c, x_2) = f(x_1 + c, x_2)$ or $f(-x_1 + 2c, x_2) = f(x_1, x_2)$, then

$$
\begin{aligned}
sgn(\varphi'_+) k_{\varphi_+}\left(\varphi_-(x_1 - c), x_2\right)\left|\varphi'_-(x_1 - c)\right| &= f(x_1, x_2), \\
sgn(\varphi'_-) k_{\varphi_-}\left(\varphi_+(x_1 - c), x_2\right)\left|\varphi'_+(x_1 - c)\right| &= f(x_1, x_2),
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$$\begin{aligned} sgn(\varphi'_+)k_{\varphi_+}\left(\varphi_-(x_1 - c), x_2\right)\left|\varphi'_-(x_1 - c)\right| &= f(x_1, x_2), \\ sgn(\varphi'_-)k_{\varphi_-}\left(\varphi_+(x_1 - c), x_2\right)\left|\varphi'_+(x_1 - c)\right| &= f(x_1, x_2), \end{aligned}$$

- If $supp\left(f\left(x_1, x_2\right)\right) \cap supp\left(f\left(-x_1 + 2c, x_2\right)\right) = \emptyset$, then on $supp\left(f\left(x_1, x_2\right)\right)$

$$\begin{aligned} sgn(\varphi'_+)k_{\varphi_+}\left(\varphi_-(x_1 - c), x_2\right)\left|\varphi'_-(x_1 - c)\right| &= f\left(-x_1 + 2c, x_2\right) = 0, \\ sgn(\varphi'_-)k_{\varphi_-}\left(\varphi_+(x_1 - c), x_2\right)\left|\varphi'_+(x_1 - c)\right| &= f\left(-x_1 + 2c, x_2\right) = 0, \end{aligned}$$

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Corollary

Let φ be an even function which satisfies the assumptions given in Case 2.
(i) If $f \in C^\infty(\mathbb{R}^2)$ satisfy $f(-x_1 + c, x_2) = f(x_1 + c, x_2)$ and have compact support in $\mathbb{R} \setminus \{c\} \times \mathbb{R}$, we have

$$f(x_1, x_2) = \frac{|\varphi'(x_1 - c)|}{4\pi} \int_{\mathbb{R}} \mathcal{H}\left(\partial_u \mathcal{R}_\varphi f\right)(s, x_2 - s\varphi(x_1 - c))\, ds. \tag{14}$$

(ii) If $f \in C^\infty(\mathbb{R}^2)$ have compact support in $\mathbb{R} \setminus \{c\} \times \mathbb{R}$ and $\operatorname{supp}(f(x_1, x_2)) \cap \operatorname{supp}(f(-x_1 + 2c, x_2)) = \emptyset$, on $\operatorname{supp}(f(x_1, x_2))$ we have

$$f(x_1, x_2) = \frac{|\varphi'(x_1 - c)|}{2\pi} \int_{\mathbb{R}} \mathcal{H}\left(\partial_u \mathcal{R}_\varphi f\right)(s, x_2 - s\varphi(x_1 - c))\, ds. \tag{15}$$

Remark

For even φ;

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- *If f is odd with respect to the line $x_1 = c$, i.e. $f(-x_1 + c, x_2) = -f(x_1 + c, x_2)$, then $\mathcal{R}_\varphi f$ is equal to zero.*

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- *If f is not odd or even with respect to the line $x_1 = c$, let us write f as a sum of its even and odd parts with respect to the line $x_1 = c$,*

$$f(x_1, x_2) = f_{even}(x_1, x_2) + f_{odd}(x_1, x_2), \tag{16}$$

Then

$$\mathcal{R}_\varphi f(s, u) = \mathcal{R}_\varphi f_{even}(s, u). \tag{17}$$

Moreover, for any odd function $g(x_1, x_2)$ with respect to the line $x_1 = c$ we have

$$\mathcal{R}_\varphi (f_{even} + g)(s, u) = \mathcal{R}_\varphi f(s, u),$$

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- *Inversion?*

Some consequences III

- If f is odd with respect to the line $x_1 = c$, then $\partial_{x_1} f$ will be even with respect to the line $x_1 = c$,

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- If f is odd with respect to the line $x_1 = c$, then $\partial_{x_1} f$ will be even with respect to the line $x_1 = c$,
- The equality

$$f(x_1, x_2) = f_{even}(x_1, x_2) + f_{odd}(x_1, x_2), \tag{18}$$

can be written as

$$f(x_1, x_2) = f_{even}(x_1, x_2) + \int_0^{x_1} \partial_{x_1}(f_{odd})(x_1, x_2) dx_1, \tag{19}$$

where

$$\partial_{x_1}(f_{odd})(x_1, x_2) = \frac{\partial_{x_1} f(x_1, x_2) + \partial_{x_1} f(-x_1 + 2c, x_2)}{2}$$

is even with respect to the line $x_1 = c$, so from (14) we have

Some consequences III

φ'_- and φ'_+ have the opposite sign

If the GRT of the partial derivative $\partial_{x_1} f$ is known;

Corollary

Let φ be an even function which satisfies the assumptions given in Case 2.
(i) If $f \in C^\infty(\mathbb{R}^2)$ satisfy $f(-x_1 + c, x_2) = -f(x_1 + c, x_2)$ and have compact support in $\mathbb{R}\setminus\{c\} \times \mathbb{R}$, we have

$$f(x_1, x_2) = \frac{1}{4\pi}\int_0^{x_1} |\varphi'(x_1 - c)| \int_{\mathbb{R}} \mathcal{H}\left(\partial_u \mathcal{R}_\varphi \partial_{x_1} f\right)(s, x_2 - s\varphi(x_1 - c))\, ds dx_1 . \tag{20}$$

(ii) If $f \in C^\infty(\mathbb{R}^2)$ and have compact support in $\mathbb{R}\setminus\{c\} \times \mathbb{R}$, we have

$$f(x_1, x_2) = \frac{|\varphi'(x_1 - c)|}{4\pi}\int_{\mathbb{R}} \mathcal{H}\left(\partial_u \mathcal{R}_\varphi f\right)(s, x_2 - s\varphi(x_1 - c))\, ds + \frac{1}{4\pi}\int_0^{x_1} |\varphi'(x_1 - c)| \int_{\mathbb{R}} \mathcal{H}\left(\partial_u \mathcal{R}_\varphi \partial_{x_1} f\right)(s, x_2 - s\varphi(x_1 - c))\, ds$$

Numerical reconstruction

- We reconstruct f from

$$\mathcal{R}_\varphi f(-\cot\theta, t\csc\theta) = \int_{\mathbb{R}} f(x_1, t\csc\theta - \cot\theta\varphi(x_1 - c))dx_1, \quad (22)$$

for several choices of φ.

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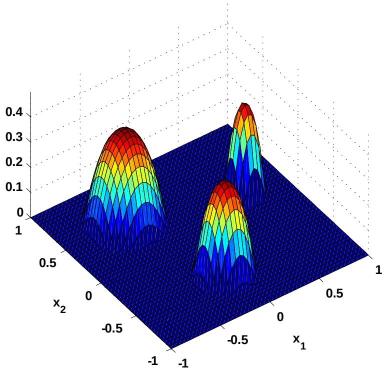
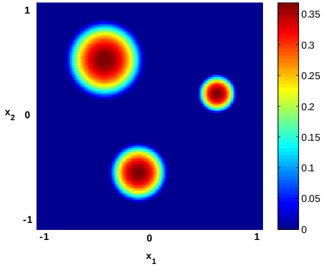
  for several choices of φ.
- We consider f defined by the sum of the functions

$$C_i(x_1,x_2) = \begin{cases} \exp\left(\dfrac{-r_i^2}{r_i^2 - (x_1-a_i)^2 - (x_2-b_i)^2}\right), \\ \quad \textit{if} \ \ (x_1-a_i)^2 - (x_2-b_i)^2 < r_i^2, \\ 0, \quad \textit{otherwise}, \end{cases} \quad (23)$$

An example of f

Parameters for $C_i(x_1, x_2)$			
i	1	2	3
Center (a_i, b_i)	$(0.6, 0.2)$	$(-0.4, 0.5)$	$(-0.1, -0.5)$
Radius r_i	0.2	0.4	0.3





Reconstruction procedure I

Reconstructing $f(x_1, x_2)$ from its integrals $\mathcal{R}_\varphi f$ consist of the following steps.

- **Step 1.** *Compute $R_\varphi f$ in (22).*

  - In the implementations, the integrals are approximated by the trapezoidal rule with the built-in function "trapz.m" of MATLAB.

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- **Step 2.** *Compute* Rk_φ *for* φ *as in Case 1 or* $R\left(sgn(\varphi'_-)k_{\varphi_-} + sgn(\varphi'_+)k_{\varphi_+}\right)$ *for* φ *as in Case 2 from* $R_\varphi f$ *obtained in Step 1.*

  - The relations between GRT and RT in Theorem 1 are used.

Reconstruction procedure II

- **Step 3.** *Reconstruct* k_φ *from* Rk_φ *or* k_{φ_-} *and* k_{φ_+} *from* $R\left(sgn(\varphi'_-)k_{\varphi_-} + sgn(\varphi'_+)k_{\varphi_+}\right)$ *obtained in Step 2.*

  - The implementations are conducted using the filtered backprojection algorithm "iradon.m" of the Image Processing Toolbox of MATLAB.

Reconstruction procedure II

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- **Step 4.** *Reconstruct* f *from* k_φ *or* k_{φ_-} *and* k_{φ_+} obtained in Step 3.

  - The relations in Lemma 1 are used.

Settings

- In our implementations
  - We set $c = 0$,
  - f is supported within the square $[-1, 1] \times [-1, 1]$,
  - k_φ or k_{φ_-} and k_{φ_+} in Lemma 1 are also supported within the square $[-1, 1] \times [-1, 1]$ due to choices of φ.

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  - We set $c = 0$,
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- We used;
  - 256×256 uniform grid for the square $[-1, 1] \times [-1, 1]$ and the discretized values of f at each node of this grid.
  - The uniformly discretized values of θ at in $(0, \pi) \cup (\pi, 2\pi)$, and t in $[-\sqrt{2}, \sqrt{2}]$.

Recontruction Example I

(φ is strictly monotonic)

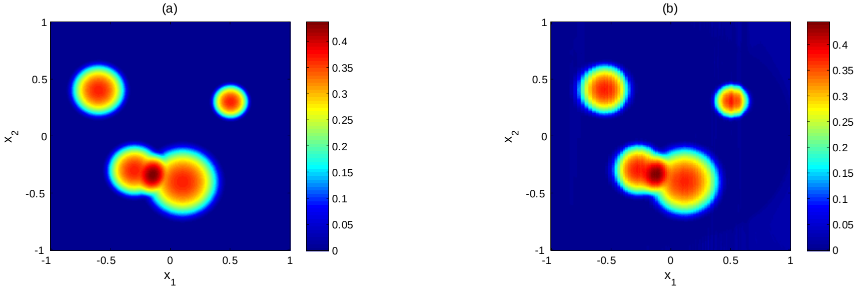

Figure: (a) Exact f, (b) reconstructed f (for $\varphi(x_1) = \exp(x_1 - 1)$)

Recontruction Example II

(φ is even f is even with respect to the line $x_1 = 0$)

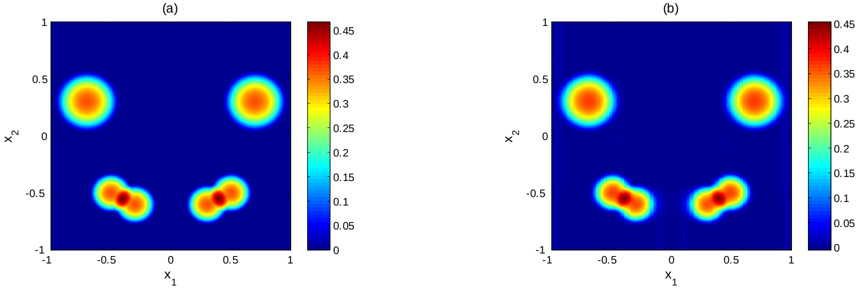

Figure: (a) Exact f, (b) reconstructed f (for $\varphi(x_1) = \dfrac{1}{1+x_1^2}$)

Reconstruction Example III

(φ is even, without the evenness condition on f)

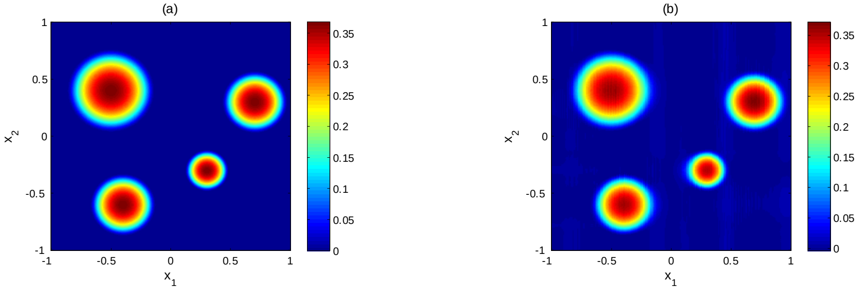

Figure: (a) Exact f, (b) reconstructed f (for $\varphi(x_1) = \dfrac{1}{1+x_1^2}$)

Here, we follow the Steps 1-4 of the inversion procedure to both f and its partial derivative $\partial_{x_1} f$.

Thanks...

THANK YOU FOR YOUR ATTENTION