Bound State Spectrum of Schrödinger Equation with Point Interactions in Hyperbolic Manifolds

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Outline

- Some basic facts in quantum mechanics
- The Schrödinger equation with Dirac delta potentials in $\mathbb{R}$
- Analysis and Heat kernel on Riemannian Manifolds
- Introduction of the Model on Riemannian Manifolds[1]
- Bound State Spectrum of the Problem, Main result on the Number of Bound States in $\mathbb{H}^3$

[1] Joint work with Teoman Turgut

Some Basic Facts in Quantum Mechanics

- Every quantum system is described by the Schrödinger operator (or Hamiltonian operator):

$$H\psi(x) := -\frac{\hbar^2}{2m}\Delta\psi(x) + V(x)\psi(x)$$

acting on $L^2(\mathbb{R}^D)$. Dirac's notation: $H\psi(x) = \langle x|H|\psi\rangle$. V is called the potential.

- $\psi \in L^2$ is called the wave function (or state vector).

- IVP for the Schrödinger equation:

$$i\frac{\partial}{\partial t}\psi = H\psi$$

and

$$\psi|_{t=0} = \psi_0 \in L^2(\mathbb{R}^D)$$

- $\int_\Omega |\psi(x,t)|^2 \, dx$: the probability that a particle is in the region $\Omega \subset \mathbb{R}^D$ at time t. The conservation of total probability:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^D} |\psi(x,t)|^2 \, dx = 0.$$

Self-adjointness of H

- The solution to IVP conserves the probability

$$\Updownarrow$$

H is symmetric

($\langle H\psi, \phi\rangle = \langle \psi, H\phi\rangle$ for all $\phi, \psi \in D(H)$).

- H is self-adjoint ($H = H^*$, i.e., H is symmetric and $D(H) = D(H^*)$) $\Longleftrightarrow$ there exists a unique solution to IVP, conserving probability, given by

$$\psi(t) = U(t)\psi_0 = e^{-itH}\psi_0$$

for all $t > 0$ (H generates one-parameter group of unitary operators, the time evolution).

- Self-adjoint $\Rightarrow$ Symmetric

$\not\Leftarrow$

Self-adjointness of H

- B is an extension of A if $D(A) \subseteq D(B)$ and $A\psi = B\psi$ for all $\psi \in D(A)$.

- Self-adjoint extension of a symmetric A:

$$A \subseteq B = B^*$$

  Can we find a self-adjoint extension of symmetric A?

- Theorem: a symmetric operator A has a self-adjoint extension if its defect indices

$$d_+ = \dim(\ker(A^* - i))$$

  and

$$d_- = \dim(\ker(A^* + i))$$

  coincide.

- How to construct? : Von Neumann's theory of self-adjoint extensions of symmetric operators.

The Spectrum of a Linear Operator

$$A : D(A) \subset L^2 \longrightarrow L^2$$

- A is invertible if there is a bounded operator A^{-1} such that $AA^{-1} = 1_{L^2}$ and $A^{-1}A = 1_{D(A)}$.

- The spectrum of A, $\sigma(A)$, is the set of all points $z \in \mathbb{C}$ for which $A - z$ is not invertible.

- The resolvent set of A, $\rho(A)$, is the set of all points $z \in \mathbb{C}$ for which $A - z$ is invertible.

- If $z \in \rho(H)$, then $(A - z)^{-1}$ is called the resolvent of A at z, and written as

$$R_A(z) = R(z) := (A - z)^{-1}$$

The Spectrum of H

- A rough classification:

  1) Discrete (point) spectrum $\sigma_p(H)$: the set of isolated eigenvalues of H with finite multiplicity.

  2) Continuous (essential) spectrum
  $$\sigma_{ess}(H) := \sigma(H) \setminus \sigma_p(H)$$

  Ex: $\sigma_{ess}(-\Delta) = [0, \infty)$ and $\sigma_p(-\Delta) = \emptyset$.

- Such a classification is related to the space-time behaviour of $\psi(x, t)$:

  Bound states (L^2 solutions, localized for all time) are related to $\sigma_p(H)$.

- There are a few exactly solvable potentials, where we can find the spectrum explicitly.

  Ex: Free particle ($V = 0$), Harmonic oscillator ($V = \frac{1}{2}kx^2$), Infinite well, so on.

A Formal Treatment of Dirac Delta Potential in $\mathbb{R}$

Bound state problem: It is enough to solve the eigenvalue equation

$$H\psi = E\psi$$

Consider the potential

$$V = -\lambda\delta(x-a),$$

where $\lambda > 0$. Treat this singular distribution as if it was a function.

$$\left[-\frac{d^2}{dx^2} - \lambda\text{``}\delta(x-a)\text{''}\right]\psi(x) = E\psi(x)\ .$$

Integrate this from $a-\epsilon$ to $a+\epsilon$:

$$-\psi'(a+\epsilon) + \psi'(a-\epsilon) - \lambda\psi(a) = E\int_{a-\epsilon}^{a+\epsilon}\psi(x)dx$$

A Formal Treatment of Dirac Delta Potential in $\mathbb{R}$

and take $\epsilon \to 0^+$:

$$\psi'(a+) - \psi'(a-) = -\lambda\psi(a)$$

Solve $H\psi = E\psi$ in each regions ($x < a$ and $x > a$) and then impose the matching conditions at $x = a$, we find the bound state energy (one bound state)

$$E = -\frac{\lambda^2}{4}$$

and the associated wave function:

$$\psi(x) = \sqrt{\lambda/2}\exp(-\lambda|x|/2)$$

⚠ Does the operator

$$H = -\frac{d^2}{dx^2} - \lambda\delta(x-a)$$

make sense mathematically?

A Rigorous Treatment of Dirac Delta Potential in $\mathbb{R}$

- Start with [2]

$$\dot{H}_a = -\frac{d^2}{dx^2}$$

with

$$D(\dot{H}_a) = \{f \in H^{2,2}(\mathbb{R}) | f(a) = 0\}$$

for some $a \in \mathbb{R}$.

All self-adjoint extensions (Von Neumann):

$$-\Delta_{\lambda,a} = -\frac{d^2}{dx^2}$$

$$D(-\Delta_{\lambda,a}) = \{\psi \in H^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R} \setminus \{a\}) | \psi'(a+) - \psi'(a-) = -\lambda\psi(a)\}$$

[2] S. Albeverio, F. Gesztesy, R. Hegh-Krohn and H. Holden, Solvable Models in Quantum Mechanics, AMS (2004)

- Formal Treatment $\equiv$ Rigorous Treatment. Why?

  Start with Dirac sequence δ_ϵ as a potential, find the resolvent of this Hamiltonian. It converges to the resolvent of the Hamiltonian constructed by self-adjoint extension method.

- The formal treatment of Dirac delta potentials in $\mathbb{R}^2$ and $\mathbb{R}^3$ fails. Infinities. Regularization and renormalization methods [3].

- OUR GOAL:

  Geometry $\Leftrightarrow$ the spectrum of such Hamiltonians

  1) Construct the Dirac delta potentials on Riemannian manifolds by renormalization techniques. Natural tool: HEAT KERNEL

  2) Search for the sufficient condition for the Hamiltonian to have the same number of bound states as the number of Dirac delta potentials.

[3] R. Jackiw, Delta-Function Potentials in Two- and Three-Dimensional Quantum Mechanics, World Scientific (1991)

Analysis on Riemannian Manifolds

- Let (M, g) a Riemannian manifold without boundary.

$$ds^2 = \sum_{i,j=1}^{D} g_{ij} dx^i dx^j$$

- Laplacian on (M, g) in a local chart (U, x):

$$\Delta_g = \frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^{D} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det(g)} \frac{\partial}{\partial x^j} \right)$$

- Integration of $f : U \to \mathbb{R}$ on a local chart (U, x)

$$\int_U f \, dV = \int_{x(U)} f(x^{-1}) \sqrt{\det(g(x^{-1}))} \, dx^1 \dots dx^D$$

Analysis on Riemannian Manifolds

Thanks to the partition of unity, integration of $f : M \to \mathbb{R}$ on M:

$$\int_M f dV = \sum_\alpha \int_{U_\alpha} \phi_\alpha \, f \, dV$$

Roughly:

$$\int_M f dV = \int_M f \underbrace{\sqrt{\det(g)}\, dx^1 \ldots dx^D}_{Riemannian\ volume\ form}$$

- ASSUME: (M, g) without boundary and Cartan-Hadamard(C-H) manifolds with Ricci tensor bounded below. In particular, hyperbolic manifolds.

Heat Equation and Heat Kernel on Manifolds

- IVP of the heat equation on Riemannian manifold (M, g)
$$\frac{\partial u}{\partial t} = \Delta_g u$$
with
$$u(x, 0) = f(x)$$

- A fundamental solution of the heat equation on M is a continuous function $K_t(x, y)$, defined on $M \times M \times (0, \infty)$, which is C^2 with respect to x, C^1 with respect to t, and which satisfies
$$\frac{\partial K_t(x, y)}{\partial t} = \Delta_g K_t(x, y)$$
and
$$\lim_{t \to 0^+} K_t(., y) = \delta_g(., y)$$

Heat Kernel

Here $\delta_g(., y)$ is the Dirac delta function, that is, for all bounded continuous functions f on M we have for every $y \in M$

$$\lim_{t \to 0^+} \int_M K_t(x, y) f(y)\, dV(y) = f(x)$$

For (M, g) without boundary[4]

$$K_t(x, x) \sim \frac{1}{(4\pi t)^{D/2}} \sum_{k=0}^{\infty} u_k(x, x)\, t^k$$

as $t \to 0^+$ for any point x. Here $u_k(x, x)$ are scalar polynomials in the curvature tensor of the manifold and its covariant derivatives at point x.

[4] P. B. Gilkey, Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem (1984).

Explicit Heat Kernel Formulas

Heat Kernel on $\mathbb{R}^D$:

$$K_t(x,y) = \frac{1}{(4\pi t)^{D/2}}\, e^{-\frac{|x-y|^2}{4t}}$$

The heat kernel on the hyperbolic manifolds $\mathbb{H}^D_\kappa$ of constant negative sectional curvature $-\kappa^2$: [5]

$$K_t^\kappa(d(x,y)) = \begin{cases} \dfrac{\sqrt{2}}{\kappa}\dfrac{1}{(4\pi t)^{3/2}}\, e^{-\kappa^2 t/4} \displaystyle\int_{\kappa d(x,y)}^{\infty} \frac{s\, e^{-s^2/4\kappa^2 t}}{\sqrt{\cosh s - \cosh \kappa d(x,y)}}\, ds\,, & \text{for } D=2 \\ \dfrac{\kappa d(x,y)}{(4\pi t)^{3/2}\sinh \kappa d(x,y)}\, e^{-\kappa^2 t}\, e^{-\frac{d(x,y)^2}{4t}}\,, & \text{for } D=3\,. \end{cases}$$

[5] A. Grigoryan, Heat Kernel and Analysis on Manifolds, AMS/IP Studies in Advanced Mathematics Vol. 47, edited by S.-T. Yau, (2009)

Bounds on the Heat Kernel

For Cartan-Hadamard manifolds [6]:

$$K_t(x,y) \leq \frac{C_1}{t^{D/2}} \exp\left(-\frac{d^2(x,y)}{C_2 t}\right)$$

for all $x, y \in M$ and $t > 0$.

Lemma. [Cheeger, Yau] [7] *If the Riemannian manifold is complete and has a Ricci tensor bounded from below, i.e.,* $\mathrm{Ric}(.,.) \geq -(D-1)k\ g(.,.)$*, with* $k \in \mathbb{R}$*, then*

$$K_t(x,y) \geq K_t^k(d(x,y))$$

where K_t^k *is the heat kernel of the simply connected complete manifold of constant sectional curvature* k*.*

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[6] A. Grigoryan, Heat Kernel and Analysis on Manifolds, AMS/IP Studies in Advanced Mathematics Vol. 47 (2009);
A.Grigoryan, Spectral Theory and Geometry, London Mathematical Society Lecture Notes Vol. 273, Cambridge University Press (1999)

[7] J. Cheeger and S-T. Yau, Comm. Pure Appl. Math. **34**, 465 - 480 (1981)

Point Interactions on Riemannian Manifolds

We consider [8] a single quantum mechanical particle intrinsically moving in a D dimensional (M, g) in the presence of Dirac delta interactions supported by a finite set of isolated points $a_i \in M$. Formally:

$$H\psi(x) = -\Delta_g \psi(x) - \text{“}\sum_{i=1}^{N} \lambda_i \delta_g(x, a_i)\psi(x)\text{”}$$

Define regularized Hamiltonian:

$$H_\epsilon \psi(x) = -\Delta_g \psi(x) - \sum_{j=1}^{N} \lambda_j(\epsilon) K_\epsilon(x, a_j) \int_M K_\epsilon(y, a_j)\psi(y)\, dV \ ,$$

[8] F.Erman, O.T. Turgut, J. Phys. A: Math. Theor. **43** 335204 (2010)

Point Interactions on Riemannian Manifolds

H_ϵ is self-adjoint on $D(H_0)$ (Kato-Rellich theorem - finite rank perturbations). If we choose

$$\frac{1}{\lambda_i(\epsilon)} = \int_\epsilon^\infty K_t(a_i, a_i)\; e^{-t\mu_i^2}\; dt$$

with $\mu_i > 0$, then for $\Re(z) < 0$ sufficiently large, the limit of the regularized resolvent converges to

$$\boldsymbol{R(z) f(x) = R_0(z)\; f(x) + \sum_{i,j=1}^{N} R_0(x, a_i|z)\; \left[\Phi^{-1}(z)\right]_{ij} R_0(z)\; f(a_j)}$$

where $R_0(z)\; f(x) = (-\triangle_g - z)^{-1} f(x) = \int_M R_0(x, y|z)\; f(y)\; dV$

Point Interactions on Riemannian Manifolds

$$\Phi_{ij}(z) = \begin{cases} \int_0^\infty K_t(a_i, a_i) \left(e^{-t\mu_i^2} - e^{tz} \right) dt & \text{if } \; i = j \\ -\int_0^\infty K_t(a_i, a_j) \, e^{tz} \, dt & \text{if } \; i \neq j. \end{cases} ,$$

called the principal matrix and $R_0(x, y|z) = \int_0^\infty e^{zt} K_t(x, y) \, dt$ is the free resolvent kernel. The resolvent satisfies the so-called first resolvent identity,

$$R(z_1) - R(z_2) = (z_2 - z_1) R(z_1) R(z_2)$$

• Is this resolvent of a unique densely defined closed operator? self-adjoint? ✓YES [9].

[9] C. Dogan, F. Erman and O.T. Turgut, J. Math. Phys. **53**, 043511 (2012)

Principal Matrix for Hyperbolic Manifolds

For the three dimensional hyperbolic spaces $\mathbb{H}^3_\kappa$,

$$\Phi_{ij}(z) = \frac{1}{4\pi}\left(\sqrt{\kappa^2 - z} - \sqrt{\kappa^2 + \mu_i^2}\right)\delta_{ij}$$
$$-(1-\delta_{ij})\left(\frac{\kappa \exp\left(-d(a_i, a_j)\sqrt{\kappa^2 - z}\right)}{4\pi \sinh\left(\kappa d(a_i, a_j)\right)}\right) \ .$$

Properties of the Principal Matrix

- $\Phi(z)$ for Cartan-Hadamard manifolds is a matrix-valued holomorphic function on the complex plane, where $\Re(z) < 0$.

  Proof: Essentially based on the following theorem[10] Let $t \in (0, \infty)$ and z a complex variable ranging over a domain $\mathcal{R}$. Assume that the function $f(z,t)$ satisfies the following conditions:

(i) $f(z,t)$ is a continuous function of both variables.

(ii) For fixed t, $f(z,t)$ is a holomorphic function of z.

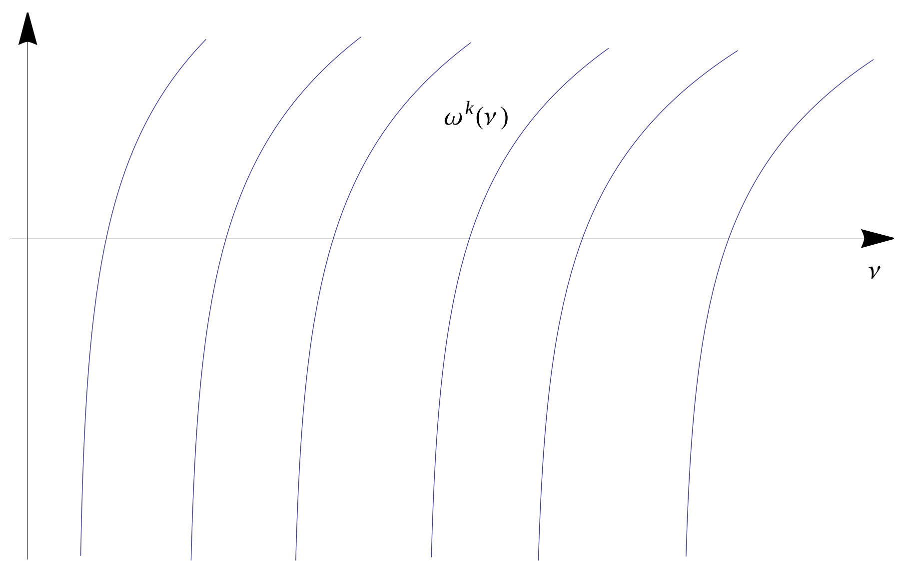
(iii) $F(z) = \int_0^\infty f(z,t)\ dt$ converges uniformly at both limits in any compact set in $\mathcal{R}$. Then, $F(z)$ is holomorphic in $\mathcal{R}$ and its derivatives of all orders may be found by differentiating under the integral sign.

[10] F. W. J. Olver, Asymptotics and Special Functions (1974)

Properties of the Principal Matrix

- Eigenvalues of the principal matrix restricted to the negative real axis are monotonic functions.

  Proof: Use previous result and Feynman-Hellmann theorem, see [11].



[11] F.Erman, O.T. Turgut, J. Phys. A: Math. Theor. **43** 335204 (2010)

Spectrum of the Problem

- Let M be Cartan-Hadamard manifold. Then, $\sigma_p(-\Delta_g) = \emptyset$ and $\sigma_{ess}(-\Delta_g) = [a, \infty)$. In particular, $a = (D-1)^2\kappa^2/4$ for $\mathbb{H}^D_\kappa$.

  ⚠ We can always shift a to the zero since it does not change the physics. Without loss of generality $\sigma_{ess}(-\Delta_g) = [0, \infty)$ for Hyperbolic manifolds.

- Using Weyl's essential spectrum theorem, we conclude that (Proof: see [12])

$$\sigma_{ess}(H) = \sigma_{ess}(-\Delta_g)$$

- Then, the point spectrum (bound state spectrum) should only come from the points z for which

$$\det \Phi(z) = 0$$

  or zeroes of the eigenvalues ω on the negative real axis

$$\Phi(z)A(z) = \omega(z)A(z)$$

[12] F. Erman, International Journal of Geometric Methods in Modern Physics, 17 (2017).

Proposition 1. [Kato] [13] *Let* $T(k) = (t_{ij}(k))_{i,j=1}^{N}$ *be a real symmetric and continuously differentiable matrix. Suppose that*

$$\lim_{k\to\infty} T(k) = diag(a_1, a_2, \ldots, a_N).$$

Then, the following holds:

There exist N *continuously differentiable functions* $\tau_i(k)$ *that represent the repeated eigenvalues of the matrix* $T(k)$ *and*

$$\lim_{k\to\infty} \tau_i(k) = a_i$$

for all $i = 1, \ldots, N$*.*

[13] T. Kato, Perturbation Theory for Linear Operators, Classics in Mathematics, Springer (1995)

Lemma. *Let*

$$T_{ij}(-\nu^2) := \frac{1}{g(-\nu^2)}\,\Phi_{ij}(-\nu^2) = \begin{cases} \dfrac{1}{\frac{1}{2\pi}\,\log \nu/\mu_i}\,\Phi_{ij}(-\nu^2)\;, & \text{for } D = 2 \\ \dfrac{1}{\frac{1}{4\pi}\,(\nu - \mu_i)}\,\Phi_{ij}(-\nu^2)\;, & \text{for } D = 3\;, \end{cases}$$

for $\nu > \mu_i$. Then, there exist N continuously differentiable functions $\omega_i(-\nu^2)/g(-\nu^2)$ that represent the eigenvalues of $T_{ij}(-\nu^2)$, where $\omega_i(-\nu^2)$ is the eigenvalue of the matrix $\Phi_{ij}(-\nu^2)$. Moreover, $\lim_{\nu\to\infty} \dfrac{\omega_i(-\nu^2)}{g(-\nu^2)} = 1$ for all i.

Proof. T is symmetric, continuously differentiable matrix for $\nu > \mu_i$.

By Lebesgue dominated convergence theorem and the upper bound of the heat kernel for CH manifolds, $\lim_{\nu\to\infty} T_{ij}(-\nu^2) = 0$ for $i \neq j$.

The diagonal lower bound for two dimensional hyperbolic manifolds of sectional curvature $-\kappa^2$ is given by [14]

$$K_t(x,x) \geq \frac{1}{8(4\pi)^{3/2}} \frac{e^{-\kappa^2 t/4}}{t\sqrt{1+\kappa^2 t}} \,,$$

for all $t > 0$ and $x \in M$.

This shows that $\Phi_{ii} \to \infty$ as $\nu \to \infty$.

[14] E. B. Davies and N. Mandouvalos, Proc. London Math. Soc. **3** 57, 182-208 (1988)

Using the short time asymptotic expansion of the diagonal heat kernel:

$$\Phi_{ii}(-\nu^2) \sim \begin{cases} \dfrac{1}{2\pi} \log \nu/\mu_i \ , & \text{for } D = 2 \\ \dfrac{1}{4\pi} (\nu - \mu_i) \ , & \text{for } D = 3 \ , \end{cases}$$

as $\nu \to \infty$. Then, $\lim_{\nu\to\infty} T_{ij}(-\nu^2) = diag(1, \dots, 1)$ so that it satisfies the hypothesis of the Theorem [Kato]. Then, the eigenvalues of the principal matrix Φ tends asymptotically to a positive function g for large values of ν. $\square$

Lemma 1. *If $\Phi(-\nu_*^2)$ is negative definite with some $\nu_* > 0$, then we have N bound states.*

Proof. Due to the previous Lemma, $\omega_i(-\nu^2) > 0$ for large enough ν. According to the assumption of the lemma, $\omega_i(-\nu_*^2) < 0$ for all i, then there exist at least N number of ν_i such that $\omega_i(-\nu_i^2) = 0$ for all i due to the IVT. Hence, it implies that $\det \Phi(-\nu_i^2) = 0$, so that $-{\nu_i}^2$ is an eigenvalue. The monotonic behaviour of ω_i's guarantees that there exists exactly N number of ν_i such that $\omega_i(-\nu_i^2) = 0$ for all i. □

Proposition 2. [Gerschgorin] [15]

All eigenvalues of a matrix T are contained in the union of Gerschgorin's disks

$$G_i = \left\{ z \in \mathbb{C}; |z - T_{ii}| \leq \sum_{j \neq i} |T_{ij}| \right\}$$

for $i = 1, \ldots, N$.

[15] R. A. Horn and C. R. Johnson, Matrix Analysis (1992).

Main Result

Let $d = \min_{1\le i,j\le N}\{d(a_i,a_j); i\neq j\}$ and

$\mu = \min_{1\le i\le N}\mu_i.$

Theorem. *(i) If there exists $\nu_* > 0$ such that*

$$\Phi_{ii}(-\nu_*^2) + \sum_{j\neq i}|\Phi_{ij}(-\nu_*^2)| < 0\ , \qquad (1)$$

then there are N bound states.

In particular,

(ii) If

$$\exp\left(d\sqrt{\kappa^2+\mu^2}-1\right)\ \left(\frac{\sinh\kappa d}{\kappa d}\right) > (N-1)\ , \quad (2)$$

holds in $\mathbb{H}^3$, then there are N bound states.

Proof. Let

$$G_i(-\nu^2) = \left[\Phi_{ii}(-\nu^2) - \sum_{j\neq i} |\Phi_{ij}(-\nu^2)|,\ \Phi_{ii}(-\nu^2) + \sum_{j\neq i} |\Phi_{ij}(-\nu^2)| \right].$$

GT $\implies$ $\omega_i(-\nu^2) \in \cup_{j=1}^{N} G_j(-\nu^2)$ for all i. Thus, $\omega_i(-\nu_*^2) < 0$ and the hypothesis of Lemma 1 holds, which then proves the statement (i).

$$\max_{1\leq i\leq N} G_i(-\nu^2) \leq \max_{1\leq i\leq N} \Phi_{ii}(-\nu^2) + (N-1) \max_{1\leq i\leq N} \max_{1\leq j\neq i\leq N} |\Phi_{ij}(-\nu^2)|.$$

For this to be negative, $\max_{1\leq i\leq N} \Phi_{ii}(-\nu^2) < 0$. For $\mathbb{H}^3_\kappa$, imposing

$$\left(\sqrt{\kappa^2+\nu^2} - \sqrt{\kappa^2+\mu^2}\right) + (N-1)\left(\frac{\kappa \exp\left(-d\sqrt{\kappa^2+\nu^2}\right)}{\sinh \kappa d}\right) < 0$$

implies the condition (i). Necessarily $\nu < \mu$. Define $F_1(\nu)$ to be the left-hand side of the above inequality. Let ν_c satisfies $F_1'(\nu_c) = 0$. Since $F_1''(\nu) > 0$, we obtain that

$$\inf_{0<\nu<\mu} F_1(\nu) = \begin{cases} F_1(\mu)\ , & \text{if } \nu_c \geq \mu \\ F_1(\nu_c)\ , & \text{if } 0 \leq \nu_c < \mu\ , \\ F_1(0)\ , & \text{otherwise} \end{cases}$$

Using this, we can see that $\inf_{0<\nu<\mu} F_1(\nu) < 0$ if and only if

$$(N-1) < \exp\left(d\sqrt{\kappa^2+\mu^2} - 1\right) \frac{\sinh \kappa d}{\kappa d}\ .$$

□

Recent Studies on the Number of Negative Eigenvalues

- Necessary and sufficient condition for the one dimensional Schrödinger operator with finitely many point δ-interactions to have the same number of negative eigenvalues as the number of point interactions in flat spaces:

  S. Albeverio and L. Nizhnik, Methods Funct. Anal. Topology **9** (4) 273-286 (2003).

  S. Albeverio and L. Nizhnik, Lett. Math. Phys. **65** 27-35 (2003).

  O. Ogurisu, Lett. Math. Phys. **85** 129 (2008).

  O. Ogurisu, Methods of Funct. Anal. Topology, **16** 4, 383 - 392 (2010).

THANK YOU FOR YOUR LISTENING